UNCLASSIFIED

AD NUMBER
ADB001641
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies only; Contractor Performance Evaluation; NOV 1974. Other requests shall be referred to U.S. Army Missile Command, Attn: AMSMI-C, Redstone Arsenal, AL 35809.**
AUTHORITY
USAMC ltr, 4 Jun 1975

THIS PAGE IS UNCLASSIFIED

REPORT RF 3248--FR 74-1 (U) AD ______________

DEVELOPMENT OF A DYNAMIC SIMULATION FILTER

FINAL REPORT

by

Gordon M. Clark
Charles B. McCartney

November, 1974

U. S. ARMY MISSILE COMMAND
Systems Analysis Division
Systems Analysis Office
Redstone Arsenal, Alabama 35809

SYSTEMS RESEARCH GROUP
Department of Industrial and Systems Engineering
The Ohio State University
Columbus, Ohio 43210

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

DEVELOPMENT OF A DYNAMIC SIMULATION FILTER

5. TYPE OF REPORT & PERIOD COVERED

Final -- June 1971 through October 1974

6. PERFORMING ORG. REPORT NUMBER

RF3248--FR 74-1(u)

7. AUTHOR(s)

Gordon M. Clark
Charles McCartney

8. CONTRACT OR GRANT NUMBER(s)

DAAH01-71-C-1258

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Systems Research Group
Department of Industrial and Systems Engineering
The Ohio State University, Columbus, Ohio 43210

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

The Ohio State University Research Foundation
1314 Kinnear Road
Columbus, Ohio 43202

12. REPORT DATE

November 1974

13. NUMBER OF PAGES

100

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

U. S. Army Missile Command
AMSMI-CS
Redstone Arsenal, Alabama 35809

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution limited to U.S. Gov't. agencies only;
Contractor Performance Evaluation; NOV 1974.
Other requests for this document must be referred to CG, USAMICOM
ATTN: AMSMI-C, Redstone, AL 35809

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Armor, Missiles, Combat Simulation, Computer Simulation, Experimental Design, Blocked Experiments, Variance Reduction, Simulation Experimental Design, Variance Estimators, Least-cost Experimental Design, Filter Model, Screening Candidates

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Filter Simulation Experimental Concept for reducing the cost of identifying a preferred system alternative by a combat simulation is investigated in this report. When using this concept, partial battles are simulated to screen candidate alternatives. The Filter concept of simulation experiments is an application of the block experimental procedure used in experimentation with physical systems. Increased correlation among alternatives improving the efficiency of the method is achieved by the use of common random number streams for all alternatives within a block. (continued)

20.

An experimental model was developed to represent effects of blocked experiments on combat simulation results, and this model includes heterogeneous variances and correlated effects. Using this model, estimators for the variance of average system performance and for the variance of the difference between the average performances of two systems were derived. Moreover, given estimates of model parameters, an algorithm was developed for determining the least-cost test plan to estimate average system performance with an upper limit being placed on the variance of the average performance of an individual system.

To test the validity of the Filter concept, four blocks of DYNCOM simulation results were analyzed. Results supported the validity of the Filter Model and indicated that experimentation costs could be reduced to one-third of the cost of traditional experimental methods.

FOREWORD

Research conducted by the Systems Research Group under Contract DAAH01-71-C-1258 is described in this report. The report is organized so that the Filter simulation experimental concept is introduced in Chapter 1, and the Filter experimental models are described in Chapter 2. Chapter 3 presents the experimental results and an analysis of these results to assess the validity and efficiency of the Filter concept. Chapter 4 presents the conclusions drawn from the research. Detailed mathematical derivations and computer program descriptions are presented in the Appendixes.

Although this is a final report for the Filter Model contract, comments, suggestions, and criticisms addressed to the authors of the report are welcome since our group maintains a continuing interest in and participation in military operations research.

Conclusions drawn in this report represent the current views of the Systems Research Group, Department of Industrial and Systems Engineering, The Ohio State University, and should not be considered as having official USAMICOM or Department of Army approval, either expressed or implied, until reviewed and evaluated by those agencies and subsequently endorsed.

The cooperation received from MICOM personnel in the conduct of this research has been extremely helpful. In particular, we wish to acknowledge the advice and assistance provided by Miss Elizabeth Watkins and Messrs. James Stage and Ernest Petty.

In addition, we would like to acknowledge the important contributions of Mr. Robert Wilhelm who wrote the Filter Model computer programs used in this research.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF TABLES

LIST OF TABLES (continued)

LIST OF FIGURES

CHAPTER 1

THE FILTER CONCEPT OF SIMULATION EXPERIMENTS

Introduction

Research conducted under contract to the United States Army Missile Command (MICOM) by The Ohio State University Systems Research Group has led to the development of DYNCOM, a high-resolution model of land combat capable of evaluating the performance of tactical units employing advanced missile systems. This report describes the results of research to develop methodology for applying DYNCOM in a more rapid manner and for reducing computer usage costs. The objective of this research is to develop a dynamic simulation filter which can screen candidates, identifying weaker alternatives, with much less cost than operating the full-scale DYNCOM simulation.

Use of Filter Models

A study of weapon design characteristics and their relationships with tactical-unit combat effectiveness often requiries the evaluation of many weapon system design alternatives and force mixes. Moreover, to adequately determine the relationships between individual weapon performance as constrained by the battlefield environment and tactical unit performance, a high-resolution simulation such as DYNCOM often is needed. This high resolution model, however, can be costly to apply if a large number of alternatives are being considered. Current simulation costs for DYNCOM are running up to $200 for a single full-length battle involving armored weapons, artillery, helicopters, and air defense weapons. To reduce the cost, lower resolution filter models and subjective analyses have been used in the past to filter candidate weapons and force mixes. An example is provided by the use in the TATAWS study of a simplified deterministic simulation called FILTER (Booz-Allen Applied Research, 1967) to screen candidate weapons and weapon mixes to reduce the work load on the more detailed IUA simulation (USACDC, 1968). However, the lack of homogeneity between filter and simulation results in the TATAWS study created severe problems in implementing this concept. That is, candidates that fared poorly in the FILTER model could do well in the IUA simulation. Moreover, this homogenity problem is always a potential problem when using a separate low-resolution filter model to screen candidates for a more detailed simulation.

The research presented in this report proposes that portions of DYNCOM itself serve as a dynamic filter when required in the context of the military problem being studied. These procedures will be similar to physical experimental design in that DYNCOM will be used to generate an environment for comparison among the alternatives being considered. This environment can be used as the starting point for further simulation experiments on each alternative. The agricultural experimenter who tests the productivity of different seed types by planting groups consisting of each type of seed at different points in a field is using a similar concept in physical experimentation. Each group of seeds has a similar environment, and the results from each group are correlated so that contrasts between the seed productivities are heightened. Each group is called a block in the language of experimental design.

Use of DYNCOM both as a fast operating filter model and as a model for a full-scale battle assures a greater degree of homogeneity between filter and simulation, and the ability to construct this filter model from the DYNCOM program takes advantage of the flexibility already inherent in the DYNCOM program. The following example illustrates the concepts described above.

Assume that a study is being conducted to evaluate the combat effectiveness of the MAW class of weapons versus the LAW and HAW classes of weapons. Further, assume that the first problem in the study is to determine the best MAW weapon, where various MAW systems have differing accuracy, lethality, and rate of fire characteristics. To start, the procedure would specify that one complete simulation of DYNCOM be made. Simulation check points would be established to determine the portion of the battle during which the different MAW candidates would be firing. A dynamic filter would thus be determined for studying the different MAW weapons by identifying the first event that a MAW weapon would be firing. Inputs to the Filter Model would be dynamic in that they would be determined from the complete simulation using DYNCOM's restart capability. One set of inputs would establish a block using the physical experimental design terminology. Replications by the dynamic filter would be made with filter inputs being modified as required, until results with the required statistical precision are obtained. Several replications in a single block may be made for each system alternative by changing the random number sequence. Moreover, identical random number sequences may be used for each replication in a given block to increase the correlation among results.

Review of Relevant Literature

Application of these concepts to a high resolution combat simulation has not been reported in the available literature. However, Fishman (1973) and Emshoff and Sisson (1970) suggest that these concepts may be profitably applied to increase the efficiency of simulation experiments. Neither authors offer any examples which show that actual simulation experiments may benefit from these precedures.

An experimental model which could be used to analyze results from blocked experiments is called the Two-Way Mixed Model (Winer, 1971; Graybill, 1961; Hicks, 1964; Scheffé, 1959; and Hocking, 1973). In this model two factors are present and many levels or possible values may exist for each factor. In this application, one factor is the combat system being analyzed and each particular system is a different level for the combat system factor. This factor is regarded as being fixed or deterministic because the different systems are predetermined and not randomly selected. The block environment is the other factor and it is regarded as being random because a very large number of possible block environments exists and each one is randomly selected from this population. Usually the assumptions are made that

1. all random effects are independent,

2. all variances are constant with respect to the system being represented, and

3. all systems have the same number of replications per block.

Scheffé (1959) and Hocking (1973) present Two-Way Mixed Models with correlations among random effects; however, the correlations proposed by their models assume homogenous variances for each system and the covariances among the replications within a block are assumed to be zero. Since data from DYNCOM suggest heterogeneous variances for each system and the experimental procedure may introduce correlations among the replications, a filter model incorporating correlations will have to be a different model than that developed by Scheffé and Hocking.

Research Required for the Filter Model

As noted in the literature review, application of the concepts of blocked experiments to simulation experiments has not been performed. Moreover, the nature of the combat process simulated by DYNCOM indicates that available Two-Factor Mixed Models may be inadequate for representing DYNCOM results. Accordingly research is required to formulate a model explicitly designed to represent DYNCOM results, and an algorithm is required for determining the least-cost test plan using this model. Then this model and other currently available models need to be tested with DYNCOM results to determine their validity. That is, the estimates of the mean and variance of system performance must be homogeneous when estimated by independent full-length DYNCOM runs and blocked filter runs.

In addition, blocked data from DYNCOM need to be analyzed to determine whether the Filter Model actually estimates systems performance more economically than independent full-length DYNCOM replications. Two types of comparisons need to be made, i.e., the Filter Model needs to be compared with independent full-length replications with respect to their efficiency in estimating mean system performance and with respect to their efficiency in estimating differences in mean performance between unlike systems. The process of filtering requires comparisons to be made between different systems so the difference in average performance between two systems is important. Finally, the ability of the Filter Model to screen candidates with small samples or modest computer expense is required to be tested to indicate the utility of the filter concept.

Organization of the Report

An overview of the three Filter Models hypothesized for use with DYNCOM is presented in Chapter 2. In addition, estimators for model parameters are presented in Chapter 2. Chapter 3 presents the analysis of DYNCOM results; a valid Filter Model is identified in the chapter, and the economies accuring from use of the Filter Model are estimated. Chapter 4 presents a summary of the research results and the principal conclusions.

Detailed mathematical derivations are presented in the appendixes. Appendix A presents the Filter Model explicitly designed to include correlations and heterogeneous variances expected to occur with DYNCOM. Estimators for the model parameters are derived in Appendix A. The least-cost experimental design algorithm is developed in Appendix B. Appendix C presents a derivation of estimators for a version of the Filter Model incorporating independent effects and heterogeneous variances. A derivation of estimators for another Filter Model version having independent effects and homogeneous variances is presented in Appendix D. Appendix E presents instructions for using the Filter program to estimate model parameters and calculate least-cost test plans.

CHAPTER 2

FILTER MODELS

Introduction

As described in Chapter 1, the basic purpose of the Filter Model is to screen alternative systems to eliminate less desirable systems using replications of homogeneous combat environments called block environments. In this chapter, several models of test results under conditions of blocking are described for consideration as the Filter Model. These models are used to predict the variance of mean system performance estimates when experiments are blocked. Moreover, an important consideration is the variance of the difference between two system performance estimates since the basic purpose of the Filter Model is to make comparisons among systems. In addition, the characteristics of estimators for model parameters required to implement the Filter Models are presented. Given parameter estimates, an algorithm specifying the least-cost test plan to reduce the variance of the mean system performance estimator below a specified upper limit is presented.

In order to more clearly present the concepts inherent in the different Filter Models, a numerical example is presented in the following section. After this example, the Filter Models are presented and compared by reference to the example.

Example of Experimenting with Blocks

Consider a combat simulation which is being used to compare the performance of armored battalions with different anti-tank weapons having dissimilar accuracy, lethality, and rate of fire characteristics. These systems are noted as System 1 and System 2. Of course, the anti-tank weapons in question are only one of several weapons in the armored battalions. The system analyst is interested in estimating performance measures in comparing these systems, and one of these performance measures is expected enemy casualties during a particular engagement.

In the simulation experimentation process, blocks are generated by simulating the initial part of the engagement until one of the two systems being compared can be employed and then generating a restart data set. This data set specifies the complete battle situation as known to the simulation, and the data set implies a block environment. The block environment consists of the state of each combatant as indicated by the following status variables:

1. casualty status; i.e., mobility kill, fire power kill, complete kill, or no kill;
2. enemy weapons detected;

3. position;
4. velocity;
5. ammunition supply;
6. fire support missions requested;
7. neutralization or suppression state; and
8. current target, etc.

Blocks are replicated by simulating the remainder of the battle until its conclusion for each system alternative and then repeating this process with a new string of random numbers starting with the same block environment.

Two options exist for performing the replications. One option consists of using a unique sequence of random numbers for each system and each replication. Another option introduces more correlation among results by using the same sequence for each alternative for a given replication. This method applies the simulation variance reduction concept of introducing positive correlation to reduce the variance of the difference between pairs of mean estimates (Emshoff and Sisson, 1970; Fishman, 1973). The latter method of using the same sequence of random numbers for each alternative is recommended.

Continuing with our example, assume that two blocks are generated, and each alternative is replicated five times in each block for a total of twenty replications or experimental results. Let

X_{ijk} = enemy casualties inflicted by the blue battalion during replication k for system j of block i.

In our example, $k = 1,2,\ldots,5$; $j = 1,2$; and $i = 1,2$.

The measure of system 1 performance would be the average of all observations; i.e., values of X_{ijk}, obtained on system 1. Let $\bar{X}_{\cdot 1 \cdot}$ be this average, then

$$\bar{X}_{\cdot 1 \cdot} = \frac{1}{10} \sum_{i=1}^{2} \sum_{k=1}^{5} X_{i1k} \ .$$

Throughout this report, average values will be noted by the symbol $\bar{X}$ with dots in place of the subscripts averaged out. In this example, assume that the enemy force has at most fifty combatant weapon systems so no value of X_{ijk} can be greater than fifty. Also using the data displayed in Table 2.1,

$$\bar{X}_{\cdot 1 \cdot} = 15.6$$

$$\bar{X}_{\cdot 2 \cdot} = 18.2 \ .$$

Table 2.1

Red Casualty Data

	Rep 1	Rep 2	Rep 3	Rep 4	Rep 5
Block 1					
System 1	15	16	17	18	15
System 2	18	20	20	17	18
Block 2					
System 1	18	17	16	18	6
System 2	18	23	17	21	10

The above system performance averages are estimates of the true, but unknown, expected number of red casualties for the two systems. Let

R_1 = expected number of red casualties for system 1

R_2 = expected number of red casualties for system 2.

If we let μ be the average value of R_1 and R_2 or

$$\mu = \frac{1}{2}(R_1 + R_2) \quad ,$$

then μ is the overall average system performance. The average of $\bar{X}_{\cdot 1 \cdot}$ and $\bar{X}_{\cdot 2 \cdot}$ is an estimator of μ; i.e.,

$$\frac{1}{2}(\bar{X}_{\cdot 1 \cdot} + \bar{X}_{\cdot 2 \cdot}) = 16.9 = \bar{X}_{\cdot\cdot\cdot}$$

is an estimate of μ. Let A_1 be the main effect due to system 1, and A_2 be the analogue of A_1 for system 2. These values are determined by

$$A_1 = R_1 - \mu \quad \text{and}$$

$$A_2 = R_2 - \mu \; .$$

Note that $A_1 + A_2 = 0$ since

$$A_1 + A_2 = R_1 - \mu + R_2 - \mu = R_1 + R_2 - 2\mu = 0 \; .$$

Also $\bar{X}_{.1.} - \bar{X}_{...} = -1.3$ is an estimate of A_1 and $\bar{X}_{.2.} - \bar{X}_{...} = 1.3$ is the corresponding estimate of A_2. The parameters μ, A_1, and A_2 will be in each Filter Model.

The values mentioned above account for differences among systems, but overlook the differences from block to block. For example, the block 1 average performance over all system replications is

$$\bar{X}_{1..} = 17.4, \text{ and}$$

$$\bar{X}_{2..} = 16.4$$

is the block 2 average performance. The above quantities are used to estimate the true block main effects, where

B_1 = true block main effect, and

B_2 = analogue of B_1 for block 2.

The block 1 expected average performance is $B_1 + \mu$, and, similarly B_2 is the block 2 expected deviation from the overall true mean performance μ. Thus, a very large number of replications of the block 1 environment over both systems would tend to give values of $\bar{X}_{1..}$ close to $B_1 + \mu$ with probabilities approaching one. Also, note that $B_1 + B_2$ does not necessarily equal zero, since the block main effects result from random selection of particular block environments from the set of all possible block environments. This is true although the estimators of B_1 and B_2 sum to zero.

In addition to the block and system effects, an interaction is likely to exist between them. Interaction is particularly likely in combat systems since casualties tend to lead to more casualties, because the force is weakened, until finally no more casualties can occur because the enemy force breaks off the engagement or becomes annihilated. The estimators of these interaction effects are

$$\left(\bar{X}_{ij.} - \bar{X}_{...}\right) - \left(\bar{X}_{.j.} - \bar{X}_{...}\right) - \left(\bar{X}_{i..} - \bar{X}_{...}\right) = \bar{X}_{ij.} - \bar{X}_{.j.} - \bar{X}_{i..} + \bar{X}_{...}$$

for $i = 1,2$ and $j = 1,2$. That is, the interaction effects estimator between block i and system j is the difference between

1. the deviation of block i and system j average performance from the overall average performance; and

2. the sum of

   a. estimated system j main effect, and
   b. estimated block i main effect.

From the data in Table 2.1, the following estimated interaction effects are generated.

Estimated Interaction Effects

Block	System 1	System 2
1	0.1	-0.1
2	-0.1	0.1

In this example, the interaction effects are smaller than the main effects; however, this result is not always true. For example, a block environment may produce large average enemy casualties by one system but not the other. These interaction effects are estimators for the true interaction effects which are noted by

AB_{ij} = true interaction effect between block i and system j.

Finally, the deviations of the individual experimental results from their respective main effects are defined as replication effects. These replication effects are noted by

ϵ_{ijk} = replication effect for replication k for system j of block i.

Collectively, the variables defined above lead to a model for experimental results which is employed by each version of the Filter Model. That is,

$$X_{ijk} = \mu + A_j + B_i + AB_{ij} + \epsilon_{ijk} \ .$$

The differences between each version of the Filter Model occur due to variations in the assumptions concerning the distributions of parameters in the above model.

Filter Model Versions

The variables μ and A_j in each version of the Filter Model are fixed or deterministic; however, B_i, AB_{ij}, and ϵ_{ijk} are stochastic or random variables since there exists a large number of possible values for B_i, AB_{ij}, and ϵ_{ijk}. Because of the manner in which B_i, AB_{ij}, and ϵ_{ijk} are defined, their mean values or expected values are zero. Three different candidate Filter Models are defined in this section, and each model is characterized by its assumption concerning the distribution of B_i, AB_{ij}, and ϵ_{ijk}. These models are listed below along with their identifying acronyms.

1. HOmogeneous Variance and Independent Effects (HOVIE),

2. HEterogeneous Variances and Independent Effects (HEVIE), and

3. HEterogeneous Variances and Correlated Effects (HEVCE).

In addition to defining each model in the following sections, the estimators for model parameters are specified.

HOVIE Model

The Homogeneous Variance and Independent Effects Model is identical to the version of the two-factor model commonly referred to as the mixed model because one factor is fixed and the other random (Hicks, 1964; Graybill, 1961; and Winer, 1971). In the HOVIE model, the random variables B_i, AB_{ij}, ϵ_{ijk}, for

$$i = 1,2,\ldots,b;$$
$$j = 1,2,\ldots,a; \text{ and}$$
$$k = 1,2,\ldots, n_{ij}$$

are all mutually independent, where

b = total number of blocks,
a = total number of system alternatives, and
n_{ij} = number of replications for system j in block i.

The variances for these random variables are

$$V(B_i) = \sigma_B^2,$$

$$V(AB_{ij}) = \sigma_{AB}^2, \text{ and}$$

$$V(\epsilon_{ijk}) = \sigma_e^2 .$$

Note that the variance for each system is identical to other systems.

The basic purposes of experimentation with simulation is to estimate the mean performance of individual systems and differences between the performance of individual systems. For system j, the average performance is

$$\bar{X}_{\cdot j \cdot} = \frac{1}{b} \sum_{i=1}^{b} \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} X_{ijk},$$

and $\bar{X}_{\cdot j \cdot}$ is an unbiased estimator of $\mu + A_j$ or the system j mean performance. A preferred experimental procedure would generate values of $\bar{X}_{\cdot j \cdot}$ with the least possible cost for nominal variances of $\bar{X}_{\cdot j \cdot}$. The variance of $\bar{X}_{\cdot j \cdot}$ is shown in Appendix D, equation D.2, to be

$$V(\bar{X}_{\cdot j \cdot}) = \frac{1}{b}\left(\sigma_B^2 + \sigma_{AB}^2\right) + \frac{\sigma_e^2}{b} \sum_{i=1}^{b} \frac{1}{n_{ij}} \qquad \text{(D.2)}$$

Note that $V(X_{\cdot j \cdot})$ is inversely proportional to b, and one term containing the variance of the replication effects is reduced by increasing the number of replications. If the cost of adding replications is not excessive and the value of σ_e^2 is significantly larger than $\sigma_A^2 + \sigma_B^2$, then multiple replications should be made with each block. On the other hand, large values of $\sigma_A^2 + \sigma_{AB}^2$ would indicate that only one replication of each block should be made.

To apply equation D.2, estimates of the parameters $\sigma_B^2 + \sigma_{AB}^2$ and σ_e^2 must be generated from available data. These estimators are noted as $\hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2$ and $\hat{\sigma}_e^2$, and are

$$\hat{\sigma}_e^2 = \sum_{i=1}^{b} \sum_{j=1}^{a} \sum_{k=1}^{n_{ij}} \left(X_{ijk} - \bar{X}_{ij\cdot}\right)^2 \Big/ \left[\sum_{i=1}^{b} \sum_{k=1}^{a} n_{ij} - ab\right] \qquad \text{(D. 6)}$$

$$\hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2 = \sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2 \;/\; (b-1)$$

$$+ \sum_{i=1}^{b} \sum_{j=1}^{a} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^2/[(a-1)(b-1)]$$

$$- \frac{(a+1)}{a^2 b} \hat{\sigma}_e^2 \sum_{j=1}^{a} \sum_{i=1}^{b} \frac{1}{n_{ij}} \qquad \text{(D. 7)}$$

where

$$\bar{X}_{ij\cdot} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} X_{ijk},$$

$$\bar{X}_{i\cdot\cdot} = \frac{1}{a} \sum_{j=1}^{a} \bar{X}_{ij\cdot}\ , \text{ and}$$

$$\bar{X}_{\cdot\cdot\cdot} = \frac{1}{b} \sum_{i=1}^{b} \bar{X}_{i\cdot\cdot}\ .$$

These estimators are unbiased which means that the expected value of the estimator is equal to the parameter value estimated; however, a practical problem may occur in implementing the estimator $\hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2$. The quantities σ_B^2 and σ_{AB}^2 are variances which implies that they are never negative, but, the statistic $\hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2$ may assume negative values by chance. When this occurs, the estimator is known to be incorrect; thus, the estimate is certainly improved by

$$\text{if } \hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2 < 0,$$

$$\text{set } \hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2 = 0.$$

The assumption is made that the true but unknown value of $\sigma_B^2 + \sigma_{AB}^2$ is probably a small positive number when $\hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2$ is less than zero, so a zero value for $\sigma_B^2 + \sigma_{AB}^2$ is fairly close to the actual value. However, this correction for negative values changes the statistic to a biased statistic since the resulting value is either equal to or greater than the unbiased statistic.

Another important estimator is the difference between two system performances, particularly when simulation experiments are being conducted in order to filter out weaker or less desirable candidate systems. The estimator for the difference between system j and t's performance is

$$\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot} \quad .$$

The efficiency with which filtering occurs is directly related to the magnitude of the variance of this difference. Since all random variables in the HOVIE Model are assumed to be mutually independent, then the variance of the difference is simply

$$V(\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot}) = V(\bar{X}_{\cdot j \cdot}) + V(\bar{X}_{\cdot t \cdot}) \quad .$$

From a statistical viewpoint, then the filtering process is more inefficient than estimating the absolute magnitude of individual systems. The analyst should be aware that a ranking procedure based on the values of $\bar{X}_{\cdot j \cdot}$; $j = 1,2,\cdots,a$; is actually a number of pairwise comparisons or differences.

HEVIE Model

The Heterogeneous Variances and Independent Effects Model extends the HOVIE Model to account for the systems with unequal variances. Other analyses of DYNCOM output data clearly indicate that variances of the same performance measure are unequal when the system is changed. In this model, the variances of B_i and AB_{ij} are still σ_B^2 and σ_{AB}^2, respectively, but the variance of the replication variation is now

$$V(\epsilon_{ijk}) = \sigma_{ej}^2 \quad , \qquad j = 1,2,\cdots,a \quad .$$

The variance of the system average performance, $\bar{X}_{\cdot j \cdot}$ is derived in Appendix C and is

$$V(\bar{X}_{\cdot j \cdot}) = \frac{1}{b}\left(\sigma_B^{\,2} + \sigma_{AB}^{\,2}\right) + \frac{\sigma_{ej}^{\,2}}{b^2} \sum_{i=1}^{b} \frac{1}{n_{ij}} \qquad \text{(C.1)}$$

which is identical with the corresponding expression for the HOVIE Model except $\sigma_e^{\,2}$ is replaced with $\sigma_{ej}^{\,2}$. However, in actual use, the allowance of different values for $\sigma_{ej}^{\,2}$ for each system means the least cost test plan will specify different numbers of replications for each system. For the HOVIE Model, the system test plans will all be identical because the variances are identical.

To use equation C.1 estimators for $\sigma_{ej}^{\,2}$ and $\sigma_B^{\,2} + \sigma_{AB}^{\,2}$ are required, and the unbiased estimator derived in Appendix C for $\sigma_{ej}^{\,2}$ is

$$\hat{\sigma}_{ej}^{\,2} = \left[\sum_{i=1}^{b} \sum_{k=1}^{n_{ij}} (X_{ijk} - X_{ij\cdot})^2\right] \Bigg/ \left[\sum_{i=1}^{b} (n_{ij} - b)\right] \qquad \text{(C.5)}$$

Note that the estimator for $\sigma_{ej}^{\,2}$ is the same as the homogeneous estimator except that the data are partitioned and only results for a given value of j are used to estimate $\sigma_{ej}^{\,2}$. Thus, the gain in realism with this change is at the expense of efficiency in estimating values. The estimator for $\sigma_B^{\,2} + \sigma_{AB}^{\,2}$ derived in Appendix C is

$$\hat{\sigma}_B^{\,2} + \hat{\sigma}_{AB}^{\,2} = \sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2 / (b-1)$$

$$+ \sum_{i=1}^{b} \sum_{j=1}^{a} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^2 / [(a-1)(b-1)]$$

$$- \frac{(a+1)}{a^2 b} \sum_{j=1}^{a} \hat{\sigma}_{ej}^{\,2} \sum_{i=1}^{b} \frac{1}{n_{ij}}. \qquad \text{(C.14)}$$

If $\hat{\sigma}_B^{\,2} + \hat{\sigma}_{AB}^{\,2} < 0$, then set $\hat{\sigma}_B^{\,2} + \hat{\sigma}_{AB}^{\,2}$ to zero for the same reasons as used for the HOVIE Model.

Also, since the random variables in the HEVIE Model are assumed to be independent, the variance of the difference between two system performance averages is

$$V(\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot}) = V(\bar{X}_{\cdot j \cdot}) + V(\bar{X}_{\cdot t \cdot}) \quad .$$

HEVCE Model

The Heterogeneous Variance and Correlated Effects Model includes the heterogeneous variances of the HEVIE Model, but also considers potentially important correlations in the results. The significance of these correlations are explained by referring to the example given earlier in this chapter.

The influence of correlations starts after a block environment (implying a value of B_i for that block) is obtained. Assume, for example, that a particular block environment tends to produce more casualties than the overall mean value of μ (estimated by $\bar{X}_{\cdots} = 16.9$). If so, then $B_i > 0$ and, for example, could be 2.0. Given this block environment would the values of AB_{ij}; $j = 1,2,\cdots,a$; tend to be greater than zero or less than zero on the average? When the performance measure is casualties in many systems the value of the block environment is likely to affect the interaction term so that $B_i > 0$ tends to produce larger values of AB_{ij}, and $B_i < 0$ tends to reduce the value of AB_{ij}. For example, the block environment increasing block casualties by 2.0 out of a total possible of 50 may increase the interaction values by 25%. The covariance* between B_i and AB_{ij} indicates the degree of this relationship and σ_{BAB} = covariance between B_i and AB_{ij}. Note that this covariance value is assumed to be the same for all systems.

Since B and each one of the interaction terms, AB_{ij}; $j = 1,2,\cdots,a$; are correlated, then the individual AB_{ij} values within a block are likely to be correlated. The covariance* between two values of AB_{ij} for different systems of the same block is

$$\sigma_{ABAB} = \text{covariance between } AB_{ij} \text{ and } AB_{it}$$

where $t \neq j$.

In addition to correlation among the block and interaction effects, these effects may be correlated with the replication effects, ϵ_{ijk}. To illustrate this point, assume the following values for the true parameters

$$\begin{array}{llll} \mu & = 16 & A_2 & = -1 \\ B_1 & = 2 & AB_{11} & = 0.3 \\ A_1 & = 1 & AB_{12} & = -0.3 \, , \end{array}$$

where the system performance measure is in enemy casualties out of a total enemy force of 50 weapons. If the mean enemy casualties produced

*The covariance between two zero mean random variables is defined as the expected value of their product.

by this block environment for system 1 is 19.5, then correlations are present among the positive block effect of $B_1 = 2$, the interaction effect of $AB_{11} = .3$, and the replication variations ϵ_{11k}. Moreover, this correlation produces a net effect of .2 casualties since independent zero mean values of ϵ_{11k} would result in a mean number of casualties for this block environment of 19.3. Note that the overall mean for each value of ϵ_{ijk} continues to be zero, although the expected value of ϵ_{ijk} given values of AB_{ij} and B_i is probably not zero. Covariances resulting from these correlations are defined below for the HEVCE Model.

σ_{Bej} = covariance between the block i main effect and a replication effect for system j during the ith block,

σ_{ABej} = covariance between the interaction main effect for system j and a replication effect for the same system during the same block,

σ'_{ABej} = covariance between the interaction main effect for system t and a replication effect for system j during the same block,

for $j = 1,2,\cdots,a$; $t = 1,2,\cdots,a$; and $t \neq j$. Note that the covariances defined above have unique values assigned for the system producing the replication effect, and these unique values are assigned because the replication effects have heterogeneous variances.

Since the replication effects are correlated with the system-block interaction effects and the block effects, the replication effects are likely to be autocorrelated. Thus, there will be correlations among different replication effects of the same system as well as correlations between replication effects of different systems. The covariances for these replication effects are defined below.

σ'_{ej} = covariance between two different replication effects of system j;

σ'_{ejt} = covariance between the replication effects on the same replication of two different systems, i.e., j and t;

σ^*_{ejt} = covariance between replication effects on different replications of different systems, i.e., j and t.

Note that for replication effects of different systems a distinction is made as to whether the same replication is used or a different replication is involved. This distinction is made because identical random number sequences may be used for each system during the same replication. By defining the covariances in this way, the positive correlations are considered among results for identical replications of different systems.

A general relationship in the HEVCE Model applies to correlations and covariances among variables in different blocks. All random variables in different blocks are considered to be independent; thus, their covariances are zero.

Expressions for important variances and estimators of model parameters are derived in Appendix A for the HEVCE Model. The variance of the average performance for system j is given by equation

$$\overline{V}(X_{.j.}) = \frac{1}{b_j}\left(\sigma_B^2 + \sigma_{AB}^2 + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} + \sigma'_{ej}\right)$$

$$+ \frac{1}{b_j^2}\left(\sigma_{ej}^2 - \sigma'_{ej}\right) \sum_{\substack{i \\ n_{ij}>0}} \frac{1}{n_{ij}}, \qquad \text{(A.42)}$$

where

b_j = number of blocks having replications for system j, i.e., number of blocks where $n_{ij} > 0$.

Parameter estimation for applying equation A.42 is simplified considerably by estimating the following quantities in aggregate form rather than their individual components:

1. V_{bj} = system j block variance

$= \sigma_B^2 + \sigma_{AB}^2 + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} + \sigma'_{ej}$, and

2. V_{rj} = system j replication variance

$= \sigma_{ej}^2 - \sigma'_{ej}$.

Both parameter values defined above are heterogeneous in the sense that each system has unique values.

An unbiased estimator for V_{rj} is identical with the estimator for σ_{ej}^2 in the HEVIE Model defined earlier. The estimator for V_{rj} is

$$\hat{V}_{rj} = \left[\sum_{\substack{i, \\ n_{ij}>0}} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij.})^2\right] \Bigg/ \left[\sum_{\substack{i, \\ n_{ij}>0}} n_{ij} - b_j\right] \qquad \text{(A.47)}$$

However, the estimator for V_{bj} is not equal to the estimator for $\sigma_B^2 + \sigma_{AB}^2$ in either the HOMIE or HEVIE Models. If it were, the models would be equivalent from an operational viewpoint. The estimator for V_{bj}, derived in Appendix A, is

$$\hat{V}_{bj} = \frac{b_j}{(b_j - 1)\left(\sum\limits_{\substack{i,\\ n_{ij}>0}} n_{ij}\right)} \sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\,(\bar{X}_{ij\cdot} - \bar{X}_{\cdot j\cdot})^2$$

$$- \frac{1}{(b_j - 1)} \hat{V}_{rj}\left[b_j(b_j - 2)/\left(\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\right) + \frac{1}{b_j}\left(\sum_{\substack{s,\\ n_{sj}>0}} \frac{1}{n_{sj}}\right)\right] \tag{A.52}$$

If $\hat{V}_{bj} < 0$, set $\hat{V}_{bj} = 0$.

Besides using a different statistic, the estimator for V_{bj} only uses data observed for system j.

The principal application for the Filter Model is to compare system performances efficiently; thus, the variance of the difference in two system averages performances is important. The expression for this variance is

$$V(\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdot t\cdot}) = \frac{1}{b_{jt}}\Big(2\sigma^2_{AB} + 2\sigma_{ABej} + 2\sigma_{ABet} + \sigma'_{ej} + \sigma'_{et} - 2\sigma_{ABAB} - 2\sigma'_{ABej}$$

$$- 2\sigma'_{ABet} - 2\sigma^*_{ejt}\Big)$$

$$+ \frac{1}{b_{jt}^2} V_{rj} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}} + \frac{1}{b_{jt}^2} V_{rt} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{it}}$$

$$- \frac{2(\sigma'_{ejt} - \sigma^*_{ejt})}{b_{jt}^2} \sum_{\substack{i,\\ n_{ij}\cdot n_{it}>0}} \frac{1}{\max(n_{ij}, n_{it})} \tag{A.63}$$

where

b_{jt} = number of blocks having replicatins for both systems j and t, and the assumption is made that

$$b_j = b_t = b_{jt} \ .$$

Also, estimation is faciltated by aggregating parameters so that

$$V_{djt} = 2\sigma^2_{AB} + 2\sigma_{ABej} + 2\sigma_{ABet} + \sigma'_{ej} + \sigma'_{et} - 2\sigma_{ABAB} - 2\sigma'_{ABej}$$

$$- 2\sigma'_{ABet} - 2\sigma^*_{ejt}, \text{ and}$$

$$V'_{rjt} = \sigma'_{ejt} - \sigma^*_{ejt}$$

The potential significance of increasing correlation to reduce the variance of differences in average performance by using identical random number sequences is apparent from equation A.63. The identical random number sequences should increase the values of V'_{rjt} and positive values of V'_{rjt} reduce the $V(X_{\cdot j \cdot} - X_{\cdot t \cdot})$. Unbiased estimators for V_{djt} and V'_{rjt} are given in Appendix A by equations A.69 and A.67, respectively.

Least Cost Experimental Design

An algorithm is presented in Appendix B for calculating the least-cost experimental design to reduce the estimated variance of each average system performance below a specified upper limit, viz., V_s. Since the estimated replication and block variances may be different for each system, then the desired experimental design may specify different replication numbers for each system. The algorithm can be initialized with inputs specifying that some experiments may already have been performed; thus, the algorithm must determine:

1. additional replications that should be observed for each system in each block previously generated;

2. additional blocks that should be generated and the replication numbers for each new block for each system.

Two costs are considered by the algorithm; i.e.,

C_b = cost to generate a new block, and

C_r = cost to perform a single replication for one system after the block has been generated.

An interesting aspect of the problem to find the least-cost experimental design is that the cost of generating a new block benefits all systems, whereas the cost of an individual replication only contributes to reducing the variance for one system.

CHAPTER 3

EXPERIMENTAL RESULTS

Introduction

A set of DYNCOM experiments were generated in order to indicate the potential for the Filter Model and test the concepts presented in Chapter 2. The resulting experimental data are analyzed in this chapter in order to address the following issues:

1. Which Filter Model is the most valid and useful representation of DYNCOM results?

2. Does the Filter concept of simulation experimentation permit significant reductions in experimental cost to achieve a specified variance of system average performance?

3. Does the Filter concept of simulation experimentation permit significant reductions in experimental cost to achieve a specified variance of the difference between the average performance of two systems?

4. Can the Filter Model be applied to screen candidates system alternatives using a low cost set of DYNCOM runs?

A total of forty-six DYNCOM simulated battles were conducted in June 1972 to answer the questions posed above. The forces involved consisted of four blue Armored Personnel Carrier (APC) Weapons in defensive positions, and thirty-one attacking red tanks. The red tanks were equipped with conventional main guns, and the defending blue APC's were armed with an anti-tank missile. The red tanks were organized into three maneuver units of approximately company size consisting of eleven, ten, and ten tanks.

Two system tactical alternatives were simulated to represent differences in system effects. System one restricted the attacking reds from opening fire until they closed within 1200 meters of the blues, and system two represented a red opening fire range of 800 meters. Because of these opening fire ranges, a block environment had to occur before a red would open fire when employing the 1200 meter tactic. Since the blues could open fire earlier than the reds, firing and casualties could and did occur before the block environment.

Experiments with four blocks were conducted. The block environment was generated by simulating a complete battle and recording the complete DYNCOM data set and status variables at the time of the block environment during the course of the run. Actually, the block environments were selected after the fact because the battle generating the block environment would output a data set and status variables periodically during the course of the battle. Because of this preselected period, the block environments could have been selected more efficiently by dynamically determining the first red firing event under the 1200 meter opening fire tactic and then recording the data set and status variables just prior to this first red firing event.

After obtaining a block environment by the procedure described above, then the block was replicated for each system. To increase the correlation among systems within each block, a common random number sequence was employed for each system on a given replication. Each block consisted of five replications of system one and four of system two.

Thirty-six of the simulated battles were conducted to replicate the four blocks described above, and ten additional full-length battles were simulated using system one to serve as a basis for comparison. That is, the variance of an individual battle could be estimated by the Filter Models from blocked data, and this estimate could be compared with the corresponding estimated variance using the ten independent battle observations.

Four performance measures that have been employed by MICOM in actual studies are used in the analyses described in this chapter. These system performance measures are:

1. red casualties
2. rounds fired by red weapons at blue weapons
3. blue exposure time, and
4. first engagement range.

More explicit definitions employed for these measures as they were applied in this study appear below. A casualty must include loss of firepower capability, so a complete kill, a mobility and firepower kill, and a firepower-only kill would be counted as a casualty. On the other hand, mobility-only kills would not be included. The rounds-fired performance measure was the total rounds fired by the red tank main gun. Blue exposure time consists of the cumulative time that an individual blue weapon had been acquired by red weapons. For example, if blue 1 had been acquired by red 1 for 50 seconds and red 2 for 75 seconds, then the exposure time for blue 1 is 125 seconds. An acquisition implies sufficient knowledge to concentrate fire upon the acquired weapon, and this acquisition can be accomplished either by direct visual contact or by pinpointing. The blue exposure time performance measure was computed for an individual battle by averaging the exposure time for each blue weapon. The first engagement range is the range at which red weapons open fire at the blue weapons. This range was computed for each blue weapon by determining the maximum range that any red weapon fired at a particular blue. In the event no reds fired at a particular blue, the minimum range that the blue fired at any red was taken as the red opening fire range so long as the red was permitted to open fire. Once these ranges were determined for each blue weapon, they were averaged to determine the first engagement range for a red.

The system performance measures from each replication of the forty-six DYNCOM replications are displayed in Tables 3.1 through 3.5.

TABLE 3.1

Red Casualty Data from Blocked Simulation Replications

		Red Casualties			
Replication	System	Block 1	Block 2	Block 3	Block 4
1	1	12	11	15	18
1	2	16	21	18	18
2	1	12	13	16	17
2	2	13	10	20	23
3	1	12	16	17	16
3	2	19	15	20	17
4	1	17	18	18	18
4	2	21	20	17	21
5	1	15	11	15	6

TABLE 3.2

Rounds Fired at Blue Weapons Data from Blocked Simulation Replications

Replication	System	Rounds Fired at Blue Block 1	Block 2	Block 3	Block 4
1	1	17	17	8	17
1	2	9	0	4	4
2	1	15	13	15	14
2	2	5	4	6	2
3	1	15	7	18	21
3	2	4	4	3	6
4	1	19	11	21	8
4	2	8	0	10	3
5	1	8	15	20	6

TABLE 3.3

Blue Exposure Time Data from Blocked Simulation Replications

Replication	System	Blue Exposure Time in Seconds Block 1	Block 2	Block 3	Block 4
1	1	201.07	404.77	369.73	347.44
1	2	277.88	52.50	151.38	152.30
2	1	325.10	429.99	705.33	327.74
2	2	117.69	111.13	216.19	76.225
3	1	366.08	186.75	501.91	427.94
3	2	72.498	155.53	167.96	122.37
4	1	454.72	354.03	536.45	197.05
4	2	148.07	127.50	175.96	110.17
5	1	610.94	156.09	637.37	203.77

TABLE 3.4

Engagement Range Data from Blocked Simulation Replications

Replication	System	Engagement Range in Meters Block 1	Block 2	Block 3	Block 4
1	1	1135.5	1148.0	1046.0	1100.8
1	2	768.25	780.3	739.75	725.5
2	1	1064.8	1145.3	1151.8	1124.8
2	2	719.75	756.5	714.50	749.25
3	1	1127.8	1100.0	1134.5	1078.5
3	2	767.75	712.0	717.25	733.5
4	1	1076.3	1029.0	1156.0	1016.0
4	2	752.75	783.8	695.5	791.00
5	1	899.50	1157.8	1118.5	1068.5

TABLE 3.5

System 1 Performance Data from Independent
Full Length Simulation Replications

Replication	Red Casualties	Rounds Fired	Blue Exposure Time (s)	Engagement Range (m)
1	13	13	397.29	1089.00
2	16	15	509.14	1144.25
3	11	15	156.09	1157.75
4	14	6	159.71	1045.50
5	16	5	218.46	1018.25
6	7	13	142.71	1187.00
7	14	6	97.01	1078.75
8	13	12	351.04	1123.75
9	15	20	637.46	1160.00
10	10	8	190.42	1062.50

Validity of the Filter Models

To test the validity of the Filter Models, estimates of the variance of an individual simulation result were computed from the blocked data using each version of the Filter Model, and these estimates are compared with an estimate of the corresponding variance from the ten independent full-length simulation replications.

If

Y_i = performance measure for the ith simulation replication from the full-length independent simulated battles,

then an estimate of the variance of Y_i is

$$\hat{V}(Y_i) = \frac{1}{9}\sum_{i=1}^{10} Y_i^2 - \left(\sum_{i=1}^{10} Y_i\right)^2 / 90$$

To estimate the variance of X_{ijk} from the blocked data, the equations for the $V(\bar{X}_{\cdot j \cdot})$ are used for the special case of one block having one replication. Of course, estimates for each parameter were computed using the estimators shown in Chapter 2. That is, the following estimating equations for $V(X_{ijk})$ were employed.

HOVIE Model

$$\hat{V}(X_{ijk}) = \hat{\sigma}^2_B + \hat{\sigma}^2_{AB} + \hat{\sigma}^2_e$$

HEVIE Model

$$\hat{V}(X_{ijk}) = \hat{\sigma}^2_B + \hat{\sigma}^2_{AB} + \hat{\sigma}^2_{ej}$$

HEVCE Model

$$\hat{V}(X_{ijk}) = \hat{V}_{bj} + \hat{V}_{rj}$$

Only values for system one could be compared because the independent full-length simulation replications were only performed for that system.

Tables 3.6 through 3.9 present comparisons computed for the variance estimate of the three candidate Filter Models. Although the variance estimates from the independent full-length simulation replications are used as the standard for comparison, recall that there are only ten of these independent observations for each performance measure. To simplify the notation in these tables $\hat{\sigma}^2_B + \hat{\sigma}^2_{AB}$ is noted as $\hat{V}_{b1}$, and $\hat{\sigma}^2_e$ as $\hat{V}_{r1}$ for the HOVIE and HEVIE Models.

Inspection of these tables reveals that the variance estimates from the HEVCE model are superior to either of the other two models. The estimates from the HEVCE model are closer to the estimates from the

TABLE 3.6

Estimates of the Variance of Total Red Casualties for One Simulation Replication (1200 Meter Red Opening Fire Range)

	Independent Full-Length Replications	HOVIE Model	HEVIE Model	HEVCE Model
$\bar{X}_{\cdot 1 \cdot}$	12.9	14.65	14.65	14.65
$\hat{V}_{b1}$		-1.73	-1.79	-.685
$\hat{V}_{r1}$		11.22	10.68	10.68
$\hat{V}(X_{i1k})$	8.10	11.22	10.68	10.68

TABLE 3.7

Estimates of the Variance of Total Rounds Fired at Blue for One Simulation Replication (1200 Meter Red Opening Fire Range)

	Independent Full-Length Replications	HOVIE Model	HEVIE Model	HEVCE Model
$X_{\cdot 1 \cdot}$	11.3	14.25	14.25	14.25
$\hat{V}_{b1}$		-1.744	-.945	-1.983
$\hat{V}_{r1}$		16.52	24.5	24.5
$\hat{V}(X_{i1k})$	24.01	16.52	24.5	24.5

TABLE 3.8

Estimates of the Variance of Blue Exposure Time (s)
(1200 Meter Red Opening Fire Range)

	Independent Full-Length Replications	HOVIE Model	HEVIE Model	HEVCE Model
$\bar{X}_{\cdot 1 \cdot}$	285.9	386.8	386.8	386.8
$\hat{V}_{b1}$		5614.	6196.	10340.
$\hat{V}_{r1}$		10610.	16425.	10425.
$\hat{V}(X_{i1k})$	32520.	16224.	22620.	26770.

Table 3.9

Estimates of the Variance of Engagement Range (m)
(1200 Meter Red Opening Fire Range)

	Independent Full-Length Replications	HOVIE Model	HEVIE Model	HEVCE Model
$\bar{X}_{\cdot 1 \cdot}$	1106.7	1094.0	1094.0	1094.0
V_{b1}		4305.2	4369.8	87.722
V_{r1}		3259.2	3900.2	3900.2
$V(X_{i1k})$	3140.7	7564.9	8270.0	3987.9

independent full-length replications than the HOVIE Model for each performance measure. When comparing the HEVCE and HEVIE models, the variance estimates are tied in two cases, viz., red casualties and total rounds fired at blue. For the blue exposure time and engagement range performance measures, the HEVCE model is clearly superior. In fact, the HEVIE estimates the engagement range variance 167% larger than the estimate from the independent full-length replications whereas the HEVCE model is only 27% larger. Because the HEVCE model appears to generate significantly better variance estimates than the other two experimental models, the assumptions made in deriving the model must be more valid. Thus, the HEVCE model will be the only model analyzed in the remainder of the report.

Although, the HEVCE is preferred to the other two candidates, the larger question remains as to whether the HEVCE model is sufficiently valid to represent DYNCOM results. Further inspection of tables 3.6 through 3.9 indicates that the maximum error by the HEVCE model in estimating the variance from independent full-length simulation replications was 32%. This result corresponds with an F statistic of 1.32; however, the performance measures are not normally distributed. Nevertheless, an F statistic of 1.32 would support the validity of the HEVCE model because the F statistic would have to be greater than 3.18 to be significant at the .05 level. Moreover, the result of 1.32 is the largest of four admitedly correlated variance ratios. Thus, these results certainly support the validity of the HEVCE model as being representative of DYNCOM experimental results.

Cost of Estimating Mean System Performance

In this section, the costs of estimating mean system performance using the Filter Model concept of blocking are compared with the use of full-length independent simulation replications. To compare these experimentation costs, the number of simulated events is used as a cost measure. That is, the cost of generating a block is specified by the number of events required to produce a block environment, and the cost of a single replication of the block is measured by the number of events subsequent to the block environment to complete the simulated battle. Thus, the cost components used by the least-cost experimental design algorithm are:

C_b = the expected number of events to generate a new block

C_r = the expected number of events subsequent to the block environment to simulate a battle for one system alternative.

Using the results from four blocks and a total of thirty-six replications, these cost components are estimated to be

C_b = 249.25 events

C_r = 558.83 events.

In addition, the cost of an independent full-length simulation replication is

$C_b + C_r$ = 808.08 events.

In addition, another cost measure is the equivalent number of full-length simulation replications which is calculated by dividing the total number of events expected for an experimented plan by 808.08 events. If the reader is interested in a dollar cost measure, the full-length simulation replications of the scenario described in this chapter required from thirty to thirty-five computer service minutes on an IBM 360/65 and the current cost for MICOM of this machine is $100 per computer service hour so a full-length simulated battle would cost from $50 to $60. In an actual study the number of alternative systems requiring consideration may be in the hundreds or thousands.

Using the cost measures given above for C_b and C_r, several comparisons were made to indicate the potential for the Filter Model concept in reducing simulation experiments costs when estimating mean system performance for individual system alternatives. These comparisons assumed that the experiments would start with two blocks in order to estimate model parameters. The cost of two blocks consisting of five replications of system one and four of system two was 10557 events or 13.06 equivalent full-length simulation replications. Note that a commonly used experimental plan for DYNCOM is to replicate each alternative ten times which would cost 6.94 full-length replications more than the two blocks.

The comparisons consisted of contrasting the costs of estimating mean system performance measures with specified variances by blocked simulation experiments versus independent full-length experiments. Two methods of determining variances for comparison purposes are used:

1. Equivalent variances for both methods of experimentation.

2. Variances less than specified upper limits for both methods of experimentation.

Experimental costs for method 2 as stated above were determined using the least-cost experimental design algorithm presented in Appendix B. The effectiveness of this algorithm is related to the accuracy of its inputs, viz., the block and replication variances for each system alternative. To illustrate the effect of sample size for these inputs, experimental designs were compared based upon input variances calculated from both two and four blocks of data.

Cost Comparisons for Achieving Equivalent Variances

Tables 3.10 through 3.13 present cost comparisons for achieving equivalent variances of the average system performance measures, viz., red casualties, rounds fired at blue, blue exposure time and engagement range. Entries in these tables were calculated by:

1. estimating the block, $\hat{V}_{bj}$, and replication, $\hat{V}_{rj}$ variances using equations A.52 and A. 47, respectively, and all four blocks of data.

2. calculating $\hat{V}(\bar{X}_{\cdot j \cdot})$ the estimated variance of the average performance for system j, by inserting the estimates $\hat{V}_{bj}$ and $\hat{V}_{rj}$ into equation A.53.

3. calculating the estimated variance of a full-length simulation replication of system by

$$\hat{V}(X_{ijk}) = \hat{V}_{bj} + \hat{V}_{rj}$$

4. determining the number n_f of independent full-length simulation replications to realize a variance for the average system performance equal to $V(\bar{X}_{\cdot j \cdot})$ by

$$n_f = \frac{V(X_{ijk})}{\hat{V}(\bar{X}_{\cdot j \cdot})}$$

5. estimating the cost of independent full-length runs by $n_f \cdot 808.08$ events.

Any negative values resulting from the computational equation for $\hat{V}_{bj}$ are shown in the tables although these negative values are set to zero in steps 2 and 3 above. Note that the Filter Model parameters were estimated using all available data in step 1. This was done to provide an accurate picture of the relationship between the test design and estimated variances. However, the comparison is made with estimated variances resulting from a test plan involving only two blocks.

Results from the cost comparisons to achieve equal variance estimates are mixed. The Filter Model concept of blocking achieved red casualty variances in 13.06 equivalent full-length runs that would require 20 independent full-length runs; thus blocking permits a reduction of 35% in experimentation costs in this case. On the other hand, variances for rounds fired, blue exposure time, and engagement range would require about the same costs for each experimentation method. Thus, the Filter Model concept of blocking is likely to reduce experimentation costs for some performance measures, viz., casualties, but not all performance measures.

Table 3.10

Cost Comparison to Achieve Equivalent Variance of Average Red Casualties, Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
$\hat{V}_{bj}$	-.685	-.844
$\hat{V}_{rj}$	10.68	11.94
$\hat{V}(\bar{X}_{\cdot j \cdot})$	1.068	1.492
$\hat{V}(X_{ijk})$	10.68	11.94
Cost of independent runs	8080.1 events or 10 runs	8080.1 events or 10 runs
Total cost of independent runs	16161 events or 20 runs	
Cost of blocked runs	10559 events or 13.06 runs	

Table 3.11

Cost Comparison to Achieve Equivalent Variance of Rounds Fired at Blue, Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
$\hat{V}_{bj}$	-1.983	2.656
$\hat{V}_{rj}$	24.5	5.875
$\hat{V}(\bar{X}_{\cdot j \cdot})$	2.45	2.063
$\hat{V}(X_{ijk})$	24.5	8.531
Cost of independent runs	8080.1 events or 10 runs	3343 events or 4.13 runs
Total cost of independent runs	11423 events or 14.13 runs	
Cost of blocked runs	10557 events or 13.06 runs	

Table 3.12

Cost Comparison to Achieve Equivalent Variance of Blue Exposure Time, Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
$\hat{V}_{bj}$	10340.	301.34
$\hat{V}_{rj}$	16425.	2855.2
$\hat{V}(\bar{X}_{.j.})$	6812.5	507.57
$\hat{V}(X_{ijk})$	26765.	3156.5
Cost of independent runs	3174.9 events or 3.93 runs	5025.4 events or 6.22 runs
Total cost of independent runs	8200.3 events or 10.15 runs	
Cost of blocked runs	10557 events or 13.06 runs	

Table 3.13

Cost Comparison to Achieve Equivalent Variance of Engagement Range, Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
$\hat{V}_{bj}$	87.72	172.89
$\hat{V}_{rj}$	3900.2	697.40
$\hat{V}(\bar{X}_{.j.})$	433.88	173.62
$\hat{V}(X_{ijk})$	3987.9	870.29
Cost of independent runs	7434.6 events or 9.20 runs	4050.6 events or 5.01 runs
Total cost of independent runs	11485.2 events or 14.21 runs	
Cost of blocked runs	10557 events or 13.06 runs	

Cost Comparisons for Reducing Variances Below Specified Upper Limits

Tables 3.14 through 3.17 present the comparisons of cost to reduce the variances of average system performance below specified upper limits. The situation assumed for these comparisons involved an initial set of experiments and then additional replications or blocks as needed for each system to realize the targeted upper limit on the variance of average system performance. This initial commitment to experiments on each system is assumed to be required in order to estimate variances or verify earlier estimates. For the blocked runs the initial commitment consisted of two blocks with five replications of system one and four of system two, and the corresponding initial commitment for the independent full-length replications consisted of seven replications for system one and six of system two. The total cost in each case of the initial set of experiments was approximately 13 equivalent full-length runs.

To reduce the effect of sampling errors, all variance parameters used to construct the comparisons in Tables 3.14 through 3.17 were estimated using four blocks of simulation replications. This procedure is consistent with the comparisons made in the previous section to compare costs with equivalent variances.

The results displayed in tables 3.14 through 3.17 were calculated by the least-cost experimental design algorithm to reduce average red casualty variances below 1.00, average rounds fired variances below 1.20, exposure time variances below 3150, and engagement range variances below 350. These results permit the following observations. Experimentation costs would be decreased through the use of blocking to estimate red casualties by 25%, rounds fired by 16%, and engagement range by 12%. However, the costs appear to increase to estimate average exposure time variance. Note that the least-cost algorithm specifies that no new blocks should be generated to estimate average red casualties and engagement range; however, four new blocks should be generated to estimate average rounds fired. Moreover, the algorithm specifies additional blocks with a single replication on each system to estimate average blue exposure time. This test plan is equivalent to independent full-length simulation replications which would be less expensive in this case than replicating blocks.

An overall evaluation of the Filter concept of blocking based upon these results is that the concept permits reduction in experimentation costs depending on the performance measure considered. However, the reductions are not dramatic. As far as the exposure time case where costs increased due to blocking, this result could be avoided if previous experience indicated the inefficiency of blocking and simulation experiments were to be set up primarily to estimate exposure time.

Table 3.14

Cost Comparison to Achieve Average Red Casualties Variance Lower Than 1.00 - Additional Replications After Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
Additional replications in previous blocks	1	2
Additional blocks	0	0
Replications in additional blocks	0	0
Total cost of blocked runs	13912 events or 17.22 runs	
Cost of independent full-length runs	8889 events or 11 runs	9697 events or 12 runs
Total cost at independent full-length runs	18586 events or 23 runs	

Table 3.15

Cost Comparison to Achieve Average Rounds Fired Variance Lower Than 1.20 - Additional Replications After Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
Additional replications in previous blocks	0	0
Additional blocks	4	4
Replications in additional blocks	3	1
Total cost of blocked runs	20495 events or 25.36 runs	
Cost of independent full-length runs	16970 events or 21 runs	6465 events or 8 runs
Total cost of independent full-length runs	23434 events or 29 runs	

Table 3.16

Cost Comparison to Achieve Average Blue Exposure Time Variance Lower Than 3150 - Additional Replications After Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
Additional replications in previous blocks	0	0
Additional blocks	6	1
Replications in additional blocks	1	1
Total cost of blocked runs	15405.5 events or 19.06 runs	
Cost of independent full-length runs	7273 events or 9 runs	4848 events or 6 runs
Total cost of independent full-length runs	12121 events or 15 runs	

Table 3.17

Cost Comparison to Achieve Average Engagement Range Variance Lower Than 350 - Additional Replications After Two Blocks Versus Independent Full-Length Runs (All Parameters Estimated Using Four Blocks of Data)

	System 1 (OFR < 1200 m)	System 2 (OFR < 800 m)
Additional replications in previous blocks	2	0
Additional blocks	0	0
Replications in additional blocks	0	0
Total cost of blocked runs	12792 events or 15.83 runs	
Cost of independent full-length runs	9697 events or 12 runs	4848 events or 6 runs
Total cost of independent full-length runs	14545 events or 18 runs	

Sensitivity of Experimental Plans to Input Variability

Ideally, the simulation experimenter will have an ample data base to estimate parameters and plan blocked simulation experiments. With precise values of the block variances, V_{bj}, and replication variances, V_{rj}, the DYNCOM experiments can be designed so that least-cost experiments are always implemented. If independent full-length experiments are the least-cost strategy, then the least-cost algorithm will specify blocks with single replications.

More commonly, the values of V_{bj} and V_{rj} will have to be estimated as part of the experimental plan. The sensitivity of these parameter estimates and the resulting experimental plans to sample size are investigated in this section by comparing the results obtained from two blocks with results from four blocks (including the former two block results).

Table 3.18 presents the parameter estimates for both two and four blocks. Recall that negative values of $\hat{V}_{bj}$ will be regarded as zero values in deriving test plans. Of course, a block variance of zero has special significance in that it implies that nothing is to be gained by generating new blocks. The results for two blocks show six out of eight cases having negative estimates for the block variances; however, three of these results are changed when two more blocks are obtained. One would expect that the true values of the block variances to be greater than zero, but they are hopefully small to the point of being negligible when the estimator of $\hat{V}_{bj}$ is negative. This rationale is supported by two of the three cases where $\hat{V}_{bj}$ changes from negative to positive. That is, V_{b2} for system-two exposure time changes from a negative value to 30..34 where the value of the replication variance is 9.5 times larger or 2855.0, and the system 2 block variance estimate for engagement range changed from a negative value to 172.89 where the replication variance is four times larger or 697.40. A small block variance relative to the replication variance will result in a least-cost design with a large number of replications per block. A less desirable situation occurs in the system-one exposure time case where a negative block variance estimate changes to 10339 when four blocks are run, and this block variance is 63% as large as the replication variance of 16425.

The effects of these variations in estimated values are shown in Tables 3.19 and 3.20 where test plans are compared based on estimate from two versus four blocks of data. Each test plan is designed to reduce the variance of average system performance below the same upper limits specified in Tables 3.14 through 3.17. These upper limits are noted as V_s in Tables 3.19 and 3.20. Two sources of differences between the test plans are present in these tables:

a. change in the overall magnitude of the variance of average system performance

b. change in relationship between the block and replication variance.

Table 3.18

Comparison of the Estimates of HEVCE Parameters from Two and Four Blocks of Data

	Red Casualties		Rounds Fired at Blue	
	2 Blocks	4 Blocks	2 Blocks	4 Blocks
System 1 (OFR < 1200 m)				
$\hat{V}_{b1}$	-1.480	-.685	-.780	-1.983
$\hat{V}_{r1}$	7.500	10.675	16.000	24.500
System 2 (OFR < 800 m)				
$\hat{V}_{b2}$	-4.458	-.844	8.750	2.656
$\hat{V}_{r2}$	18.958	11.938	5.500	5.875

	Exposure Time		Engagement Range	
	2 Blocks	4 Blocks	2 Blocks	4 Blocks
System 1 (OFR < 1200 m)				
$\hat{V}_{b1}$	-130.7	10339.	331.73	87.72
$\hat{V}_{r1}$	19600.	16425.	5972.7	3900.2
System 2 (OFR < 800 m)				
$\hat{V}_{b2}$	-311.56	301.34	-183.21	172.89
$\hat{V}_{r2}$	4836.4	2855.0	804.83	697.40

Table 3.19

Comparison of Least-Cost Experimental Plans Subsequent to Two Blocks (Two Block Parameter Estimates Versus Four Block Estimates)

	Red Casualties $V_s = 1.00$		Rounds Fired at Blue $V_s = 1.20$	
	2 Blocks Estimates	4 Blocks Estimates	2 Blocks Estimates	4 Blocks Estimates
System 1 (OFR < 1200 m)				
Additional replications in previous blocks	0	1	0	0
Additional blocks	0	0	10	4
Replications in additional blocks	0	0	1	3
System 2 (OFR < 800 m)				
Additional replications in previous blocks	6	2	0	0
Additional blocks	0	0	10	4
Replications in additional blocks	0	0	1	1

Table 3.20

Comparison of Least-Cost Experimental Plans Subsequent to Two Blocks
(Two Block Parameter Estimates Versus Four Block Estimates)

	Blue Exposure Time $V_s = 3150$		Engagement Range $V_s = 350$	
	2 Blocks Estimates	4 Blocks Estimates	2 Blocks Estimates	4 Blocks Estimates
System 1 (OFR < 1200 m)				
Additional replications in previous blocks	0	0	0	2
Additional blocks	0	6	3	0
Replications in additional blocks	0	1	4	0
System 2 (OFR < 800 m)				
Additional replications in previous blocks	0	0	0	0
Additional blocks	0	1	0	0
Replications in additional blocks	0	1	0	0

Note that the major source of differences between the test plans is the magnitude of the test effort rather than the allocation of effort between blocks and replications.

Cost of Comparing Mean System Performances

Probably the most important application of DYNCOM is to compare mean system performance in order to determine which systems are superior and by what margins. Thus, the differences between average system performances may be more important to the analyst than the absolute magnitude of average system performances. The Filter concept of blocked experiments contributes to improving the efficiency of estimating the differences among system performances by increasing the correlation among results for different systems. The use of blocks with fixed block environments for each replication and the use of identical random number sequences for each alternative replicated within a block causes the desired correlation. Considering these correlations, equation A.63 gives the variance of the difference in average performance between two systems. Thus, A.63 can be used to estimate the degree of improvement, if any, in efficiency resulting from the Filter concept of blocking.

In this section, the experimental results are analyzed to answer two questions:

1. are the assumptions leading to equation A.63 for determining the variance of the difference in average performance valid for representing DYNCOM results?

2. does the Filter concept of blocked simulation experiments permit savings in estimating the variance of the difference in average performances of two systems?

Validity of the Expression for $V(\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot})$

To test the validity of equation A.63, the variance of the difference between the average performance of systems one and two was computed directly using four replications of each block, i.e., the fifth replication of system one was ignored. For each replication, the difference between system one and system two performance was calculated. That is,

Y_{ik} = difference between system one and two performance for the k^{th} replication of block i

$$Y_{ik} = X_{i1k} - X_{i2k}$$

Then the values of Y_{ik} were inserted into equations A.47 and A.52 to estimate V_{rd} and V_{bd}, the replication and block variances for differences between system performance. The subscript d is used to denote differences. Then the quantities $\hat{V}_{rd}$ and $\hat{V}_{bd}$ are substituted into equation A.42 to estimate the variance of the difference between system one and system two performance. This variance estimate is noted as $\hat{V}(\bar{Y})$.

Values of $\hat{V}(\bar{Y})$ are shown in table 3.21 and compared with estimates of $V(\bar{X}_{\cdot 1 \cdot} - \bar{X}_{\cdot 2 \cdot})$ computed from equation A.63 using all four blocks of data with the fifth replication of system one deleted. Note that $\hat{V}(\bar{Y})$ and $\hat{V}(\bar{X}_{\cdot\cdot\cdot} - \bar{X}_{\cdot 2 \cdot})$ are identical for each performance measure with the exception of engagement range. In the engagement range case, the value of $\hat{V}_b$ is estimated as -275.14 and then set to zero as specified in equation A.52. If $\hat{V}_{b1}$ retained the -275.14 value and this value was used in estimating $V(\bar{X}_{\cdot 1 \cdot} - \bar{X}_{\cdot 2 \cdot})$, then $\hat{V}(\bar{X}_{\cdot 1 \cdot} - X_{\cdot 2 \cdot})$ would become 267.14 and be identical to $\hat{V}(\bar{Y})$ for engagement range.

Thus, the close match among the two alternative methods of calculating variances of differences in average performance serves to support the validity of equation A.63. Although the engagement range case suggests that negative values of $\hat{V}_{bj}$ should be retained, this modification is not recommended because of the possibility that estimates of $\hat{V}(\bar{X}_{\cdot 1 \cdot} - \bar{X}_{\cdot 2 \cdot})$ may also become negative.

Table 3.21

Variance of the Difference in Average Performances Validity Check
(Four Blocks With Four Replications Per System Per Block)

	Red Casualties	Rounds Fired at Blue	Exposure Time	Engagement Range
$\hat{V}(\bar{Y})$	.7904	1.1745	1371.7	267.14
$\hat{V}(\bar{X}_{\cdot 1 \cdot} - \bar{X}_{\cdot 2 \cdot})$	.7904	1.1745	1371.7	335.93
$\hat{V}(\bar{X}_{\cdot 1 \cdot}) + \hat{V}(\bar{X}_{\cdot 2 \cdot})$	1.5846	2.3125	2615.2	230.59
F	.4988	.5079	.5245	1.457

Cost of Estimating Differences in Average System Performance

Besides showing estimates of the variance of the difference in average system performances, Table 3.21 computes estimates of these variances under the assumption that observations of system one and system two are independent. Under the assumption of independence, the variance of the difference in average performance is estimated by:

$$\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot}) = \hat{V}(\overline{X}_{\cdot 1 \cdot}) + \hat{V}(\overline{X}_{\cdot 2 \cdot})$$

With correlation, the $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$ becomes much smaller as shown in Table 3.21. The values of F shown in the table are the ratio between the estimates of $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$ and the estimate under the assumption of independence. Most of the F values are approximately .5 implying a 50% reduction in variance by the Filter Model concept of blocking. The only exception is engagement range, where F is approximately 1.5 suggesting larger variances of differences than independent sampling.

A better comparison between the variances of differences in performance under independent and blocked experiments is shown in Table 3.22. All of the blocked data including the fifth observation on system one are used in the variance estimates. Again, the effect of blocking permits a significant reduction in variance with exception of engagement range. For rounds fired, the variance under blocking is less than 40% of its values under independent sampling. Differences in red casualties and exposure time have variances under blocking of 50% of corresponding values under independent sampling. The engagement range case must have some negative correlation, i.e., a high engagement range in a battle where the reds can open fire at 1200 metters would tend to produce a low engagement range when the 800 meter opening fire tactic is played. However, if the analyst is primarily interested in casualties, rounds fired, and exposure time; the Filter Model concept of blocking will produce an estimated reduction in variance of 50% for a specified experimentation cost.

Table 3.22

Variance of the Difference in Average Performance
(Four Blocks With Five Replications of System One and Four of System Two)

	Red Casualties	Rounds Fired at Blue	Exposure Time	Engagement Range
$\hat{V}(\overline{X}_{\cdot 1 \cdot}) + \hat{V}(\overline{X}_{\cdot 2 \cdot})$	1.2798	2.2563	3660.0	303.75
$\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$	.6403	.8797	1874.7	436.98
F	.5003	.3899	.5122	1.439

Screening Candidate Alternatives

A primary objective for the Filter Model is to screen or filter candidate systems with low-cost replications of DYNCOM in order to eliminate weaker systems. In this section, a single block with five replications on each system is evaluated with respect to its efficiency in filtering candidate systems. Based on the experiments evaluated in this report, the cost for two systems is 7.22 equivalent full-length replications. To evaluate this filter, the variance of the difference in average casualties is computed for each performance measure, and the cost in independent full-length replications to achieve the same variance is computed.

To estimate the variance of the difference in average casualties, equation A.63 becomes $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot}) = \hat{V}_{d12} + (\hat{V}_{r1} + \hat{V}_{r2})/5 - 2\hat{V}'_{r12}/5$ for this case of one block with five replications of each system. Also, negative values of $\hat{V}_{d12}$ are set to zero in order to avoid the possibility of $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$ becoming negative. Values of $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$ are presented in Table 3.23 for each performance measure. The cost in equivalent independent full-length replications to match $\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})$ is determined by

1. computing the variance of the difference for one full-length replication on each alternative system by

$$\hat{V}(X_{11k}) + \hat{V}(X_{12k})$$

2. computing the cost to match the single block filter by

$$2 \cdot \frac{\hat{V}(X_{11k}) + \hat{V}(X_{12k})}{\hat{V}(\overline{X}_{\cdot 1 \cdot} - \overline{X}_{\cdot 2 \cdot})}$$

equivalent full-length replications.

The calculation procedure shown above assumes that each system will have the same number of replications in both the Filter and independent full-length replications.

The cost comparisons in Table 3.23 show a very large cost reduction for the Filter in comparing red casualties and rounds fired. These performance measures would require 23.08 full-length runs for casualties and 20.48 runs for rounds fired to match the Filter variance obtained with only 7.22 runs. In other words, the single block Filter is about three times more efficient than independent full-length replications for casualties and rounds fired. However, the exposure time and engagement range performance measures show about the same cost for the Filter and independent full-length replications.

Table 3.23

Efficiency of a Single Block Filter With Five Replications

	Red Casualties	Rounds Fired	Exposure Time	Engagement Range
$\hat{V}_{d12}$	5.1229	-.2938	2306.	3737.2
$\hat{V}_{r1}$	10.675	24.50	16426.	3900.2
$\hat{V}_{r2}$	11.938	5.875	2855.	697.4
$\hat{V}'_{r12}$	19.203	7.125	-2984.6	7359.3
$\hat{V}(\bar{X}_{\cdot 1 \cdot} - \bar{X}_{\cdot 2 \cdot})$	1.964	3.225	7356.0	1713.05
$\hat{V}(X_{i1k})$	10.675	24.50	26765.	3987.9
$\hat{V}(X_{i2k})$	11.938	8.53	3156.5	870.29
Cost of independent replications	23.03 runs	20.48 runs	8.14 runs	5.67 runs
Cost of filter replications	7.22 runs	7.22 runs	7.22 runs	7.22 runs

CHAPTER 4

SUMMARY AND CONCLUSIONS

In Chapter 2 several experimental design models were hypothesized for the purpose of representing the results of blocked DYNCOM runs. The objective for these models is to investigate the economics of operating DYNCOM to generate a battle situation or block environment, and then replicating the remainder of the battle for each alternative. These blocked replications would be useful if the variance in average system performance or the variance of the difference in average system performance for two systems were less for a fixed computer expense. A primary objective would be to use one or more inexpensive blocks to filter candidates to eliminate weaker alternatives. This Filter is required to be inexpensive and have small variances for the difference in average performances for two systems.

Several models were proposed because the assumptions inherent in experimental models that are available in the literature appear inappropriate for a combat model like DYNCOM. The primary extensions made to derive the Filter Experimental Model involve correlations among block, interaction, and replication effects and heterogeneous variances.

A set of DYNCOM experiments were conducted at MICOM to evaluate these Filter Models and test their efficiency. These experiments involved two system tactical alternatives where the opening fire range for the attacking red force was set at 1200 meters for system one and 800 meters for system two. Four performance measures were analyzed, viz., red casualties, rounds fired at blue weapons, blue exposure time, and first engagement range. An important characteristic of the experiment was that a common random number sequence was used for each replication in a block on each system in order to increase the correlation among the results for the two systems. By increasing correlation, the variance of the difference in average performance is reduced. The results of the analysis of the experiment are presented in Chapter 3. Of course, these results should be interpreted as illustrating the potential for the Filter Model concept of DYNCOM experimentation, and results for other systems or battlefield environments maybe different.

The overall conclusions drawn from these experiments are that the Filter Experimental Model is a valid representation of DYNCOM experimental results, and that significant economies can be realized by using DYNCOM as a filter depending upon the analysts choice of performance measures for ranking the system alternatives. Excellent results were obtained in reducing simulation costs if the performance measures of interest were red casualties or rounds fired. For example, a DYNCOM filter using one block and five replications per alternative obtained estimated variances for the difference in average red casualties or rounds fired in approximately one-third the cost of independent full-length runs. Results for blue exposure time and engagement range showed no reduction in simulation expense.

When using more than one block to compare system performances by computing the differences in system performance, the Filter concept of blocking continues to offer the simulation experimenter more for a fixed level of effort. Results obtained from four blocks indicate that the Filter experimental procedure gives a variance one-half of the variance expected on the basis of independence for red casualties, rounds fired, and exposure time. The engagement range variance was larger than expected as a result of independent experiments.

Comparisons were also made when the experimental objective was to estimate the mean value of the performance measure. The best result for the Filter Model was a 35% reduction in cost to estimate expected red casualties with equivalent variances from both blocked filter runs and independent full-length simulation runs. A least-cost experimental design algorithm is presented in Appendix B, and this algorithm determines the number of replications and blocks for each system in reducing the variance of average performance for each system below a specified level. Results from this algorithm showed a 25% reduction in cost to estimate expected red casualties, a 16% reduction for rounds fired, and a 12% reduction for engagement range. The costs for exposure time estimation appeared to increase. Thus, the savings to estimate average system performance are less dramatic than estimating differences in average system performance, but the savings for estimating expected casualties are significant. For the other performance measures, the potential for savings in unclear.

APPENDIX A

DERIVATION OF ESTIMATORS FOR FILTER MODEL PARAMETERS (HEVCE Model)

Introduction

In this appendix, the version of the Filter Model that incorporates heterogeneous variances and correlated effects is defined and unbiased estimators are derived for parameter values that are important to the simulation experimenter. That is, certain parameter values are required to plan further simulation experimentation and to estimate the

(a) mean performance of each system alternative,
(b) difference between mean performance of a pair of alternatives,
(c) variance of the mean performance estimators, and
(d) variance of difference estimators.

The model acronym is HEVCE for HEterogeneous Variances and Correlated Effects. The Filter Model is specifically designed to represent important characteristics of DYNCOM, a stochastic combat simulation, likely to be encountered when simulation is performed in blocks. That is, a given battle situation or environment is obtained, and a block of replications are observed (or simulated) on each alternative. This starting situation for a block is called a block environment. Given a particular block environment differences among the system designs may be more efficiently observed since a source of variation has been removed. Of course, the subsequent results may be related to the particular block environment so a number of blocks may be required to reliably estimate system performance measures.

Filter Model Structure

The structure of the Filter Model is defined by a set of assumptions concerning variation among system alternatives, block effects, stochastic differences from replication to replication, and interactions among these effects. For a given replication, the following equation is assumed by the Filter Model to relate the principal effects. That is,

$$X_{ijk} = \mu + B_i + A_j + AB_{ij} + \epsilon_{ijk} \quad ; \qquad \text{(A.1)}$$

where

X_{ijk} = observed system performance for block i, system j, and replication k;

μ = mean effect over all systems, blocks, and replications;

B_i = main effect for block i;

A_j = main effect for system j;

AB_{ij} = interaction effect between block i and system j;

ϵ_{ijk} = replication effect for replication k of system j during block i.

There are b blocks so $i = 1,2,\cdots,b$ and a different system alternatives so $j = 1,2,\cdots,a$. In addition, n_{ij} replications are simulated for block i of system j. Thus, $k = 1,2,\cdots,n_{ij}$.

The stochastic properties of the above variables and their mutual relationships are important in being able to represent the results of different test plans. The usual experimental design models assume that the stochastic variables are mutually independent and normally distributed (Hicks, 1964, Graybill, 1961, and Winer, 1971). Significant extensions to these assumptions are made in the HEVCE Model. To represent blocking effects in DYNCOM, the following assumptions are made. The parameters μ and A_j; $j = 1,2,\cdots,a$; are fixed effects or deterministic. That is, the mean performance for system j is $\mu + A_j$. Since μ is the mean performance overall alternatives,

$$\sum_{j=1}^{a} A_j = 0. \tag{A.2}$$

The block main effects, B_i, $i = 1,2,\cdots,b$, are random since the set of b blocks are selected from a large number of different possible blocks. Each value of B_i is assumed to be independently and identically distributed with mean and variance given by

$$E(B_i) = 0 \tag{A.3}$$

$$V(B_i) = \sigma_B^2 \tag{A.4}$$

The block-system interaction effects, AB_{ij}, are also assumed to be identically distributed random variables with mean and variance

$$E(AB_{ij}) = 0 \tag{A.5}$$

$$V(AB_{ij}) = \sigma_{AB}^2 \tag{A.6}$$

Since the system levels are fixed or deterministic, then

$$\sum_{j=1}^{a} AB_{ij} = 0 \quad \text{for all } i \tag{A.7}$$

but the block levels are stochastic so

$$\sum_{i=1}^{b} AB_{ij} \neq 0 \quad \text{for all } j. \tag{A.8}$$

In addition, the block and interaction effects for a given block are correlated with covariances

$$COV(B_i, AB_{ij}) = E(B_i \cdot AB_{ij}) = \sigma_{BAB}; \tag{A.9}$$

for

$$j = 1,2,\cdots,a$$
$$i = 1,2,\cdots,b$$

also

$$COV(AB_{ij}, AB_{it}) = E(AB_{ij} \cdot AB_{it}) = \sigma_{ABAB}; \tag{A.10}$$

for

$$j = 1,2,\cdots,a$$
$$i = 1,2,\cdots,b$$

Since the results among different blocks are mutually independent, the

$$COV(B_v, AB_{ij}) = E(B_v \cdot AB_{ij}) = 0, \tag{A.11}$$

$$COV(AB_{ij}, AB_{vj}) = E(AB_{ij} \cdot AB_{vj}) = 0, \tag{A.12}$$

$$COV(AB_{it}, AB_{vj}) = E(AB_{it} \cdot AB_{vj}) = 0, \tag{A.13}$$

for

$$i \neq v; \quad i = 1,2,\cdots b;$$
$$v = 1,2,\cdots b;$$

$$t \neq j; \quad j = 1,2,\cdots a; \text{ and}$$
$$t = 1,2,\cdots a.$$

The replication effects, ϵ_{ijk}, have unique distributions for each system alternative. Their means and variances are

$$E(\epsilon_{ijk}) = 0, \tag{A.14}$$

$$V(\epsilon_{ijk}) = \sigma^2_{ej}, \tag{A.15}$$

for

$$j = 1,2,\cdots,a;$$
$$i = 1,2,\cdots,b; \text{ and}$$
$$k = 1,2,\cdots,n_{ij}.$$

These replication effects are correlated with both the block and interaction effects for the block being replicated. That is,

$$COV(B_i, \epsilon_{ijk}) = \sigma_{Bej} \tag{A.16}$$

$$COV(AB_{ij}, \epsilon_{ijk}) = \sigma_{ABej} \tag{A.17}$$

$$COV(AB_{it}, \epsilon_{ijk}) = \sigma'_{ABej} \tag{A.18}$$

$$COV(B_v, \epsilon_{ijk}) = 0 \tag{A.19}$$

$$COV(AB_{vj}, \epsilon_{ijk}) = 0 \tag{A.20}$$

$$COV(AB_{vt}, \epsilon_{ijk}) = 0 \tag{A.21}$$

for $t \neq j$, $v \neq i$, $k = 1, 2, \ldots, n_{ij}$;

$j = 1, 2, \ldots, a$;

$t = 1, 2, \ldots, a$;

$i = 1, 2, \ldots, b$; and

$v = 1, 2, \ldots, b$.

Note that each covariance involving a replication effect either has unique values for each system being replicated or is zero. Moreover, the replication effects within a block are correlated and the covariances are related to the system being replicated. That is,

$$COV(\epsilon_{ijk}, \epsilon_{ijm}) = \sigma'_{ej}, \tag{A.22}$$

for $m \neq k$, $k=1,2,\cdots, n_{ij}$; $m = 1,2,\cdots,n_{ij}$

$$COV(\epsilon_{ijk}, \epsilon_{itk}) = \sigma'_{ejt} ; \tag{A.23}$$

for $k = 1,2,\cdots,\min(n_{ij},n_{it})$

$$COV(\epsilon_{ijk}, \epsilon_{itm}) = \sigma^*_{ejt} \tag{A.24}$$

for $m \neq k$, $k = 1,2,\cdots,n_{ij}$; $m = 1,2,\cdots n_{it}$;

$$COV(\epsilon_{ijk}, \epsilon_{vjm}) = 0 \tag{A.25}$$

for $k = 1,2,\cdots,n_{ij}$; $m = 1,2,\cdots,n_{vj}$;

$$COV(\epsilon_{ijk}, \epsilon_{vtm}) = 0 \tag{A.26}$$

for $k = 1,2,\cdots,m_{ij}$; $m = 1,2,\cdots,n_{vt}$;

where $t \neq j$,
$t = 1,2,\cdots,a$;
$j = 1,2,\cdots,a$;
$i \neq v$;
$i = 1,2,\cdots,b$; and
$v = 1,2,\cdots,b$.

The covariance σ'_{ej} represents correlation among different replications of the same system, i.e., j, and the covariance σ^*_{ejt} represents correlation among different replications of different systems, i.e., j and t. Note that σ'_{ejt} represents correlation among the same replications of different systems, i.e., j and t. Unique values for the covariances among replication effects for the same replication of different systems are assigned. This is done to represent the effects of variance reduction using identical random number sequences for each system on a given replication which tends to produce a positive correlation between system performance measures for the same replication, i.e., ϵ_{ijk} and ϵ_{itk}.

Variance of Estimator for Mean System Performance

The basic purpose of simulation experiments conducted with the process defined above is to estimate the mean performance of systems with an unbiased statistic having the least possible variance. An unbiased estimator for system mean performance is presented in this section, and then an expression for the variance of this estimator in terms of the parameters defined above is derived.

Estimator for Mean System Performance

An unbiased estimator for mean system performance of system j will be developed by determining the expected value of X_{ijk} and then computing an average value for X_{ijk} over all blocks and replications for a fixed value of j. The expected value of X_{ijk} is

$$E(X_{ijk}) = \mu + A_j \tag{A.27}$$

since

$$E(B_i + AB_{ij} + \epsilon_{ijk}) = 0$$

by equations A.3, A.5, and A.14. Note that $\mu + A_j$ is the mean performance of system j. Thus, an unbiased estimator for $\mu + A_j$ is obtained by averaging the values of X_{ijk} over all blocks and replications for a fixed value of j. The sample average for a fixed block, i.e., i, and system, i.e., j, is noted by $\bar{X}_{ij\cdot}$, where

$$\bar{X}_{ij\cdot} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} X_{ijk} \ , \tag{A.28}$$

where $n_{ij} > 0$ for $i = 1,2,\cdots,b$ and $j = 1,2,\cdots,a$. Averaging over all blocks, the unbiased estimator for system j performance is $\bar{\bar{X}}_{\cdot j \cdot}$ and is defined as

$$\bar{X}_{\cdot j \cdot} = \frac{1}{b_j} \sum_{n_{ij}>0} \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} X_{ijk} , \qquad \text{(A.29)}$$

where b_j = number of blocks having at least one replication of system j; and

$$E(\bar{X}_{\cdot j \cdot}) = \mu + A_j \qquad \text{(A.30)}$$

Going one step further, the mean of $\bar{X}_{\cdot j \cdot}$ or $\bar{X}_{\cdot\cdot\cdot}$ over all alternatives is an unbiased estimator of μ. That is,

$$\bar{X}_{\cdot\cdot\cdot} = \frac{1}{a} \sum_{j=1}^{a} \frac{1}{b_j} \sum_{\substack{i \\ n_{ij}>0}} \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} X_{ijk} \qquad \text{(A.31)}$$

and

$$E(\bar{X}_{\cdot\cdot\cdot}) = \mu \qquad \text{(A.32)}$$

because of equation A.2.

Variance of Mean System Performance Estimator

Having obtained an estimator for mean system performance, an expression for the variance of this estimator will be derived in this section. First, the variance of an individual observation or $V(X_{ijk})$ is found. In deriving both variance expressions, use will be made of the fact that

$$V(Y) = E(Y^2) - E^2(Y) \qquad \text{(A.33)}$$

for any random variable Y.

To use A.33 in finding $V(X_{ijk})$, the $E(X_{ijk}^2)$ will be determined. By definition

$$\begin{aligned} E(X_{ijk}^2) &= E[(\mu + A_j + B_i + AB_{ij} + \epsilon_{ijk})^2] \\ &= E[(\mu + A_j)^2 + B_i^2 + AB_{ij}^2 + \epsilon_{ijk}^2 + 2(\mu + A_j)(B_i + AB_{ij} + \epsilon_{ijk}) \\ &\quad + 2B_i(AB_{ij} + \epsilon_{ijk}) + 2AB_{ij}\epsilon_{ijk}] \end{aligned}$$

(A.34)

In the above expression, note that

$$E[2(\mu + A_j)(B_i + AB_{ij} + \epsilon_{ijk})] = 0$$

since the expected value of a product involving a constant and a random variable with mean zero is zero. Substituting the above expression into A.34 and using A.4, A.6, A.9, A.15, A.16, and A.17, the following expression for $E(X_{ijk}{}^2)$ is obtained.

$$E(X_{ijk}{}^2) = (\mu + A_j)^2 + \sigma^2{}_B + \sigma^2{}_{AB} + \sigma^2{}_{ej} + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} \tag{A.35}$$

Using A.35, A.33, and A.27;

$$V(X_{ijk}) = \sigma^2{}_B + \sigma^2{}_{AB} + \sigma^2{}_{ej} + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} \tag{A.36}$$

To derive an expression for $V(\overline{X}_{\cdot j \cdot})$, the mean system performance estimator, use will be made of the fact that results for a given block are independent of other block results. Thus,

$$V(\overline{X}_{\cdot j \cdot}) = \frac{1}{b_j{}^2} \sum_{\substack{i, \\ n_{ij}>0}} V(\overline{X}_{ij \cdot}) \tag{A.37}$$

Also, equation A.33 will be used in determining $V(\overline{X}_{ij\cdot})$; thus, an expression for $E[\overline{X}_{ij\cdot}{}^2]$ must be determined. By definition,

$$\begin{aligned} E[\overline{X}_{ij\cdot}{}^2] &= E\left[\left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} (\mu + A_j + B_i + AB_{ij} + \epsilon_{ijk})\right)^2\right] \\ &= E\left[\left(\mu + A_j + B_i + AB_{ij} + \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)^2\right] \\ &= E\left[(\mu + A_j)^2 + B_i{}^2 + AB_{ij}{}^2 + \left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)^2\right. \\ &\quad \left. + 2B_i\left(AB_{ij} + \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right) + 2AB_{ij}\left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)\right] \end{aligned} \tag{A.38}$$

since all cross products between a constant and random variables with mean zero vanish.

Taking the expectation of the above random variables,

$$E[\bar{X}_{ij.}{}^2] = (\mu + A_j)^2 + \sigma^2_B + \sigma^2_{AB} + E\left[\left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)^2\right]$$

$$+ 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} \qquad \text{(A.39)}$$

To evaluate A.39, the expression $E\left[\left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)^2\right]$ will be expanded; thus,

$$E\left[\left(\frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk}\right)^2\right] = \frac{1}{n_{ij}{}^2} E\left[\sum_{k=1}^{n_{ij}} \sum_{m=1}^{n_{ij}} \epsilon_{ijk} \cdot \epsilon_{ijm}\right]$$

$$= \frac{1}{n_{ij}{}^2}\left[n_{ij}\sigma^2_{ej} + (n_{ij}{}^2 - n_{ij})\sigma'_{ej}\right]$$

$$= \frac{1}{n_{ij}} \sigma^2_{ej} + (n_{ij} - 1)\sigma'_{ej}, \qquad \text{(A.40)}$$

for $n_{ij} > 0$; $i = 1,2,\cdots,b$; $j = 1,2,\cdots,a$. The above result is true since the double summation in A.40 involves n_{ij} terms where

$\epsilon_{ijk}\epsilon_{ijm} = \epsilon_{ijk}{}^2$ and $n_{ij}{}^2 - n_{ij}$ terms, where

$\epsilon_{ijk}\epsilon_{ijm}$ and $m \neq k$.

Using A.22 and A.15, equation A.40 follows. Substituting A.40 into A.39,

$$E[\bar{X}_{ij.}{}^2] = (\mu + A_j)^2 + \sigma^2_B + \sigma^2_{AB} + 2\sigma_{BAB} + 2\sigma_{Bej}$$

$$+ 2\sigma_{ABej} + \frac{1}{n_{ij}}(\sigma^2_{ej} \cdot (n_{ij} - 1)\sigma'_{ej}) \qquad \text{(A.41)}$$

for $n_{ij} > 0$; $i = 1,2,\cdots,b$; $j = 1,2,\cdots,a$. Using A.33, A.27, A.41, we obtain

$$V(\bar{X}_{.j.}) = \frac{1}{b_j{}^2} \sum_{\substack{i,\\ n_{ij}>0}} \Big[\sigma^2_B + \sigma^2_{AB} + 2\sigma_{BAB} + 2\sigma_{Bej}$$

$$+ 2\sigma_{ABej} + \frac{1}{n_{ij}}\left(\sigma^2_{ej} + (n_{ij} - 1)\sigma'_{ej}\right)\Big]$$

$$= \frac{1}{b_j}(\sigma^2_B + \sigma^2_{AB} + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} + \sigma'_{ej})$$

$$+ \frac{1}{b_j{}^2}(\sigma^2_{ej} - \sigma'_{ej}) \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}} \qquad \text{(A.42)}$$

Note that from the viewpoint of analyzing an experimental plan, the above expression for the variance of the estimator of system j mean performance can be regarded as the function of just two parameters. That is,

1. system j block variance, V_{bj}, where

$$V_{bj} = \sigma^2_B + \sigma^2_{AB} + 2\sigma_{BAB} + 2\sigma_{Bej} + 2\sigma_{ABej} + \sigma'_{ej} \qquad (A.43)$$

2. system j replication variance, V_{rj}, where

$$V_{rj} = \sigma^2_{ej} - \sigma'_{ej} \qquad (A.44)$$

Knowledge of the values of both V_{bj} and V_{rj} permits calculation of the mean value estimator variance for arbitrary values of the number of blocks and replications per block.

Estimation of the Variance of the Estimator of System Mean Performance

Estimators of the values for V_{bj} and V_{rj} are derived in this section. The required estimators are obtained by first estimating V_{rj} and then V_{bj}. Each estimator is unbiased in that they are obtained by finding a statistic with an expected value equivalent to the parameter values required.

The statistic $\sum\limits_{i,\, n_{ij}>0} \sum\limits_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij.})^2$ is used to estimate V_{rj},

and the expected value of this statistic is derived below.

$$E\left[\sum_{i,\, n_{ij}>0} \sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij.})^2\right] = E\left[\sum_{i,\, n_{ij}>0} \sum_{k=1}^{n_{ij}} (X_{ijk}^2 - 2X_{ijk}\bar{X}_{ij.} + \bar{X}_{ij.}^2)\right]$$

$$= E\left[\sum_{i,\, n_{ij}>0} \sum_{k=1}^{n_{ij}} X_{ijk}^2 - \sum_{i,\, n_{ij}>0} 2n_{ij} (\bar{X}_{ij.})^2 + \sum_{i,\, n_{ij}>0} n_{ij} (\bar{X}_{ij.})^2\right]$$

$$= E\left[\sum_{i,\, n_{ij}>0} \sum_{k=1}^{n_{ij}} X_{ijk}^2 - \sum_{i,\, n_{ij}>0} n_{ij} (\bar{X}_{ij.})^2\right] \qquad (A.45)$$

Substituting A.35, A.41, A.43, and A.44 into the above equation, we obtain

$$E\left[\sum_{\substack{i,\\ n_{ij}>0}} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij.})^2\right] = \Big(\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\big((\mu + A_j)^2 + V_{bj} + V_{rj}\big)$$

$$- \sum_{\substack{i,\\ n_{ij}>0}} \left[n_{ij}\big((\mu + A_j)^2 + V_{bj} + V_{rj}/n_{ij}\big)\right]$$

$$= V_{rj}\Big(\sum_{\substack{i,\\ n_{ij}>0}} n_{ij} - b_j\Big) \tag{A.46}$$

Thus, an unbiased estimation of V_{rj} is

$$\hat{V}_{rj} = \left(1\Big/\Big(\sum_{\substack{i,\\ n_{ij}>0}} n_{ij} - b_j\Big)\right)\left(\sum_{\substack{i,\\ n_{ij}>0}} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij.})^2\right) \tag{A.47}$$

In order to find an estimator for the parameter V_{bj}, the expected value of the statistic $\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}(\overline{X}_{ij.}-\overline{X}_{.j.})^2$ is derived below.

$$E\left[\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\big(\overline{X}_{ij.}-\overline{X}_{.j.}\big)^2\right] = E\left[\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\big(\overline{X}_{ij.}^{\,2}-2\overline{X}_{ij.}\overline{X}_{.j.} + \overline{X}_{.j.}^{\,2}\big)\right] \tag{A.48}$$

To evaluate A.48, the expectations of $\overline{X}_{ij.}\cdot\overline{X}_{.j.}$ and $\overline{X}_{.j.}^{\,2}$ are required, and expressions for these quantities are derived below.

$$E[\overline{X}_{ij.} \cdot \overline{X}_{.j.}] = \frac{1}{b_j} \sum_{\substack{s,\\ n_{sj}>0}} E[\overline{X}_{ij.}\, \overline{X}_{sj.}]$$

$$= \frac{1}{b_j} E[\overline{X}_{ij.}^{\,2}] + \frac{1}{b_j} \sum_{\substack{s,\\ n_{sj}>0\\ s\neq i}} E[\overline{X}_{ij.}\, \overline{X}_{sj.}]$$

Since the random components of X_{ijk} and X_{sjk} are independent when s and i are different blocks,

$$\bar{X}_{ij\cdot} \cdot \bar{X}_{sj\cdot} = (\mu + A_j)^2$$

after substituting A.43, A.44, and A.41 into the above expressions,

$$E\left[\bar{X}_{ij\cdot}\bar{X}_{\cdot j\cdot}\right] = (\mu + A_j)^2 + \frac{1}{b_j}\left(V_{bj} + V_{rj}/n_{ij}\right) \tag{A.49}$$

Next, the expectation of $\bar{X}_{\cdot j\cdot}^{\;2}$ is found,

$$E[\bar{X}_{\cdot j\cdot}^{\;2}] = \frac{1}{b_j} \sum_{\substack{i,\\ n_{ij}>0}} E[\bar{X}_{ij\cdot} \cdot \bar{X}_{\cdot j\cdot}]$$

$$= \frac{1}{b_j} \sum_{\substack{i,\\ n_{ij}>0}} \left((\mu + A_j)^2 + \frac{1}{b_j}(V_{bj} + V_{rj}/n_{ij})\right)$$

$$= (\mu + A_j)^2 + \frac{1}{b_j} V_{bj} + \frac{1}{b_j^2} V_{rj} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}} \tag{A.50}$$

Substituting A.41, A.43, A.44, A.49, and A.50 into A.48,

$$E\left[\sum_{\substack{i,\\ n_{ij}>0}} n_{ij}(\bar{X}_{ij\cdot} - \bar{X}_{\cdot j\cdot})^2\right] = \sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\left((\mu + A_j)^2 + V_{bj} + V_{rj}/n_{ij}\right)$$

$$- 2 \sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\left((\mu + A_j)^2 + \frac{1}{b_j}(V_{bj} + V_{rj}/n_{ij})\right)$$

$$+ \sum_{\substack{i,\\ n_{ij}>0}} n_{ij}\left((\mu + A_j)^2 + \frac{1}{b_j} V_{bj} + \frac{1}{b_j^2} V_{rj} \sum_{\substack{s,\\ n_{sj}>0}} \frac{1}{n_{sj}}\right)$$

$$E\left[\sum_{\substack{i,\\n_{ij}>0}} n_{ij}(\overline{X}_{ij\cdot} - \overline{X}_{\cdot j\cdot})^2\right] = \frac{b_j - 1}{b_j}\left(\sum_{\substack{i,\\n_{ij}>0}} n_{ij}\right)V_{bj}$$

$$+ V_{rj}\left(b_j - 2 + \frac{1}{b_j^{\,2}}\left(\sum_{\substack{i,\\n_{ij}>0}} n_{ij}\right)\left(\sum_{\substack{s,\\n_{sj}>0}} \frac{1}{n_{sj}}\right)\right) \qquad \text{(A.51)}$$

The estimator for V_{bj} is determined from A.51 by replacing V_{rj} with its estimate, which gives

$$\hat{V}_{bj} = \frac{b_j}{(b_j - 1)\left(\sum_{\substack{i,\\n_{ij}>0}} n_{ij}\right)} \sum_{\substack{i,\\n_{ij}>0}} n_{ij}(\overline{X}_{ij\cdot} - \overline{X}_{\cdot j\cdot})^2$$

$$- \frac{1}{(b_j - 1)} \hat{V}_{rj}\left[b_j(b_j - 2)\Big/\left(\sum_{\substack{i,\\n_{ij}>0}} n_{ij}\right) + \frac{1}{b_j}\left(\sum_{\substack{s,\\n_{sj}>0}} \frac{1}{n_{sj}}\right)\right] \qquad \text{(A.52)}$$

If $\hat{V}_{bj} < 0$ as computed above, set $\hat{V}_{bj} = 0$.

Since V_{bj} is a variance, a negative value for $\hat{V}_{bj}$ has no physical meaning and should be adjusted. The assumption is made that the true value of V_{bj} is a small positive number when the solution for V_{bj} is negative using equation A.51 and $\hat{V}_{rj}$ in place of V_{rj}. When this case occurs, zero is a better estimate of V_{bj} than a negative number. Although the estimates using equation A.52 are improved by this procedure, the unbiased property of the estimators lost and a small bias introduced because the resulting estimator is either equal to or greater than the unbiased estimator.

Note that the variance of the mean value estimator for system j's performance can be estimated by substituting equations A.52 and A.47 for their respective parameter values into A.42. That is,

$$\hat{V}(\overline{X}_{\cdot j\cdot}) = \frac{1}{b_j}\hat{V}_{bj} + \frac{1}{b_j^{\,2}}\hat{V}_{rj} \cdot \sum_{\substack{i,\\n_{ij}\ 0}} \frac{1}{n_{ij}} \qquad \text{(A.53)}$$

Variance of Estimator for Differences in Mean System Performance

The correlations cited above among block, interaction, and replication effects will cause the estimators, $\bar{X}_{\cdot j \cdot}$; $j = 1,2,\cdots,a$; to be mutually correlated. Thus, the variance of a difference between two system performances, e.g., $\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot}$, is not simply the sum, $V(\bar{X}_{\cdot j \cdot}) + V(\bar{X}_{\cdot t \cdot})$, as one would expect when the two estimators are independent. This characteristic is very important when comparing systems because the difference between two systems may be much more important to the analyst than their absolute magnitude. In general, positive correlation between the mean value estimators will decrease the variance of the difference, $\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot}$, below the sum, $V(\bar{X}_{\cdot j \cdot}) + V(\bar{X}_{\cdot t \cdot})$. On the other hand, the variance of the difference will be increased by negative correlation.

In this section an expression for the variance of the difference, $\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot}$, where $j \neq t$, is derived. Moreover, an unbiased estimator for the variance of the difference is derived in a manner similar to that used in the previous section.

Equation A.33 is used to determine the variance of the difference $\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot}$, thus, an expression for $E[(\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot})^2]$ is required. This expectation is derived below.

$$E[(\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot})^2] = E[\bar{X}_{\cdot j \cdot}^2 - 2\bar{X}_{\cdot j \cdot}\bar{X}_{\cdot t \cdot} + \bar{X}_{\cdot t \cdot}^2] \tag{A.54}$$

To evaluate the above equation, an expression for $E(\bar{X}_{\cdot j \cdot} \cdot \bar{X}_{\cdot t \cdot})$ is necessary, and this expression will be developed as a result of a sequence of steps involving the computation of the following expectations:

1. $E(X_{ijk} \cdot X_{itk})$,
2. $E(X_{ijk} \cdot X_{itm})$ where $m \neq k$,
3. $E(X_{ijk} \cdot \bar{X}_{it\cdot})$,
4. $E(\bar{X}_{ij\cdot} \cdot \bar{X}_{it\cdot})$,
5. $E(\bar{X}_{\cdot j \cdot} \cdot \bar{X}_{it\cdot})$, and
6. $E(\bar{X}_{\cdot j \cdot} \cdot \bar{X}_{\cdot t \cdot})$.

The first expression is

$$\begin{aligned} E[X_{ijk} \cdot X_{itk}] &= E[(\mu + A_j + B_i + AB_{ij} + \epsilon_{ijk})(\mu + A_t + B_i + AB_{it} + \epsilon_{itk})] \\ &= (\mu + A_j)(\mu + A_t) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej} \\ &\quad + \sigma_{Bet} + \sigma'_{ABet} + \sigma'_{ejt} \end{aligned} \tag{A.55}$$

after deleting all terms involving the product of a constant and a zero mean random variable and applying A.4, A.9, A.10, A.16, A.18, and A.23.

Similarly, for the second expression,

$$E[X_{ijk} \cdot X_{itm}] = E[\,(\mu + A_j + B_i + AB_{ij} + \epsilon_{ijk})\,(\mu + A_t + B_i + AB_{it} + \epsilon_{itm})\,]$$

$$= (\mu + A_t)\,(\mu + A_j) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej}$$

$$+ \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt}, \qquad \text{(A.56)}$$

where use is made of A.24.

Both A.55 and A.56 will be used in determining the third expression

$$E[X_{ijk} \cdot \overline{X}_{it\cdot}] = \frac{1}{n_{it}} \sum_{m=1}^{n_{it}} E[X_{ijk} \cdot X_{itm}]$$

$$= \frac{1}{n_{it}} \Big[(n_{it} - 1) \big((\mu + A_t)(\mu + A_j) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB}$$

$$+ \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt} \big)$$

$$+ (\mu + A_t)(\mu + A_j) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB}$$

$$+ \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma'_{ejt} \Big]$$

$$(\mu + A_t)(\mu + A_j) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej}$$

$$+ \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt} + (\sigma'_{ejt} - \sigma^*_{ejt}) / n_{it}$$

if $k \le n_{it}$. Otherwise if $k > n_{it} > 0$, then

$$E[X_{ijk} \cdot \overline{X}_{it\cdot}] = (\mu + A_j)(\mu + A_t) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB}$$

$$+ \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt} \qquad \text{(A.57)}$$

For the fourth expression,

$$E[\bar{X}_{ij\cdot} \cdot \bar{X}_{it\cdot}] = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} E[X_{ijk} \cdot \bar{X}_{it\cdot}]$$

$$= (\mu + A_j)(\mu + A_t) + \sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej}$$

$$+ \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt} + \frac{(\sigma'_{ejt} - \sigma^*_{ejt})}{\max(n_{ij}, n_{it})}, \qquad (A.58)$$

where $n_{ij}, n_{it} > 0$.

The above equation will be used in constructing the fifth expression. That is,

$$E[\bar{X}_{\cdot j\cdot} \cdot \bar{X}_{it\cdot}] = \frac{1}{b_j} \sum_{\substack{s,\\ n_{sj}>0}} E[\bar{X}_{sj\cdot} \cdot \bar{X}_{it\cdot}]$$

Recalling that all random variables in different blocks are independent,

$$E[\bar{X}_{\cdot j\cdot} \cdot \bar{X}_{it\cdot}] = \frac{1}{b_j} E[\bar{X}_{ij\cdot} \cdot \bar{X}_{it\cdot}] + \frac{\min(b_j, b_t) - 1}{b_j} (\mu + A_j)(\mu + A_t)$$

$$= (\mu + A_j)(\mu + A_t) + \frac{1}{b_j}\Bigg(\sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej}$$

$$+ \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt} + \frac{\sigma'_{ejt} - \sigma^*_{ejt}}{\max(n_{ij}, n_{it})}\Bigg),$$

if $n_{ij}, n_{it} > 0$. (A.59)

In the event either n_{ij} or n_{it} are zero,

$$E[\bar{X}_{\cdot j\cdot} \cdot \bar{X}_{it\cdot}] = (\mu + A_j)(\mu + A_t) \text{ if } b_j \text{ and } b_t > 0.$$

For the last expression,

$$E[\bar{X}_{.j.} \cdot \bar{X}_{.t.}] = \frac{1}{b_t} \sum_{\substack{i, \\ n_{it}>0}} E[\bar{X}_{.j.} \cdot \bar{X}_{it.}]$$

$$= (\mu + A_j)(\mu + A_t) + \frac{b_{jt}}{b_j b_t} (\sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB}$$

$$+ \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt})$$

$$+ \frac{1}{b_j b_t} \sum_{\substack{i, \\ n_{ij} \cdot n_{it} > 0}} \frac{\sigma'_{ejt} - \sigma^*_{ejt}}{\max(n_{ij}, n_{it})},$$

where b_{jt} = number of blocks with $n_{ij} > 0$ and $n_{it} > 0$. (A.60)

Substituting A.60 and A.50 into A.54,

$$E[(\bar{X}_{.j.} - \bar{X}_{.t.})^2] = (\mu + A_j)^2 + \frac{1}{b_j} V_{bj} + \frac{1}{b_j^2} V_{rj} \sum_{\substack{i, \\ n_{ij}>0}} \frac{1}{n_{ij}} - 2(\mu + A_j)(\mu + A_t)$$

$$- \frac{2b_{jt}}{b_j b_t} \left(\sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt}\right)$$

$$- \frac{2}{b_j b_t} \sum_{\substack{i, \\ n_{ij} \cdot n_{it}>0}} \frac{\sigma'_{ejt} - \sigma^*_{ejt}}{\max(n_{ij}, n_{it})} + (\mu + A_t)^2 + \frac{1}{b_t} V_{bt}$$

$$+ \frac{1}{b_t^2} V_{rt} \sum_{\substack{i, \\ n_{it}>0}} \frac{1}{n_{ij}}$$

$$= (A_j - A_t)^2 + \frac{1}{b_j} V_{bj} + \frac{1}{b_t} V_{bt} + \frac{1}{b_j^2} V_{rj} \sum_{\substack{i, \\ n_{ij}>0}} \frac{1}{n_{ij}}$$

$$+ \frac{1}{b_t^2} V_{rt} \sum_{\substack{i, \\ n_{it}>0}} \frac{1}{n_{it}}$$

$$- \frac{2b_{jt}}{b_j b_t} \left(\sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt}\right)$$

$$- \frac{2}{b_j b_t} \sum_{\substack{i, \\ n_{ij} \cdot n_{it}>0}} \frac{\sigma'_{ejt} - \sigma^*_{ejt}}{\max(n_{ij}, n_{it})} \qquad \text{(A.61)}$$

Using A.33, the variance of the difference becomes

$$V(\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot}) = \frac{1}{b_j} V_{bj} + \frac{1}{b_t} V_{bt} + \frac{1}{b_j^2} V_{rj} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}} + \frac{1}{b_t^2} V_{rt} \sum_{\substack{i,\\ n_{it}>0}} \frac{1}{n_{it}}$$

$$-2 \frac{b_{jt}}{b_j b_t}\left(\sigma_B^2 + 2\sigma_{BAB} + \sigma_{ABAB} + \sigma_{Bej} + \sigma'_{ABej} + \sigma_{Bet} + \sigma'_{ABet} + \sigma^*_{ejt}\right)$$

$$- \frac{2}{b_j b_t} \sum_{\substack{i,\\ n_{ij}>0,\\ n_{it}>0}} \frac{\sigma'_{ejt} - \sigma^*_{ejt}}{\max(n_{ij}, n_{it})} \qquad (A.62)$$

The above expression can be simplified considerably if the assumption is made that each block having observations for system j also contains observations for system t and vice versa. Under this assumption $b_j = b_t = b_{jt}$, and after substituting A.43 and A.44

$$V(\bar{X}_{\cdot j \cdot}- \bar{X}_{\cdot t \cdot}) = \frac{1}{b_{jt}^2}(2\sigma^2_{AB} + 2\sigma_{ABej} + 2\sigma_{ABet} + \sigma'_{ej} + \sigma'_{et} - 2\sigma_{ABAB} - 2\sigma'_{ABej}$$

$$- 2\sigma'_{ABet} - 2\sigma^*_{ejt}) + \frac{1}{b_{jt}^2} V_{rj} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}}$$

$$+ \frac{1}{b_{jt}^2} V_{rt} \sum_{\substack{i,\\ n_{it}>0}} \frac{1}{n_{it}} - \frac{2(\sigma'_{ejt} - \sigma^*_{ejt})}{b_{jt}^2} \sum_{\substack{i,\\ n_{ij}\cdot n_{it}>0}} \frac{1}{\max(n_{ij}, n_{it})}$$

when $b_j = b_t = b_{jt}$. (A.63)

Both the estimation and analysis of experimental plans are simplified if $V(\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot})$ is regarded as a function of two parameters requiring estimation in addition to V_{rj} and V_{rt}. That is, let

$$V_{djt} = 2\sigma^2_{AB} + 2\sigma_{ABej} + 2\sigma_{ABet} + \sigma'_{ej} + \sigma'_{et} - 2\sigma_{ABAB} - 2\sigma'_{ABej}$$

$$- 2\sigma'_{ABet} - 2\sigma^*_{ejt}, \text{ and} \qquad (A.64)$$

$$V'_{rjt} = \sigma'_{ejt} - \sigma^*_{ejt} \qquad (A.65)$$

V_{djt} is referred to as the block variance for the difference between system j and t, and V'_{rjt} is the replication variance for the difference between j and t.

Estimation of Replication Variance

The expected values appearing in equations A.55 and A.56 immediately suggest an unbiased estimator for the replication variance. Note that the difference

$$E[X_{ijk} \cdot X_{itk}] - E[X_{ijk} \cdot X_{itm}] = \sigma'_{ejt} - \sigma^*_{ejt} = V'_{rjt}$$

Thus, an unbiased estimate of V'_{rjt} can be constructed by taking the difference between two sample averages where one average consists of terms having the expectation equal to $E[X_{ijk} \cdot X_{itk}]$ and the other average consists of terms with an expectation of $E[X_{ijk} \cdot X_{itm}]$. The desired statistic is

$$\hat{V}'_{rjt} = \frac{1}{b_{jt}} \sum_{\substack{i,\\ n_{ij}, n_{it}>0}} \frac{1}{\min(n_{ij}, n_{it})} \sum_{k=1}^{\min(n_{ij}, n_{it})} X_{ijk} \cdot X_{itk}$$

$$- \frac{1}{b'_{jt}} \sum_{\substack{i,\\ n_{ij}\cdot n_{it}>1}} \frac{1}{n_{ij}n_{it}-\min(n_{ij}, n_{it})} \sum_{k=1}^{n_{ij}} \sum_{\substack{m=1\\ k \neq m}}^{n_{it}} X_{ijk} X_{itm}$$

where b'_{jt} is the number of blocks with $n_{ij}n_{it} > 1$; and $E[V'_{rjt}] = V'_{rjt}$.

(A.67)

Finding an estimator for V_{djt} is a more difficult task. The statistic

$$\sum_{i=1}^{b} n_{ij}\, n_{it}\, (\bar{X}_{ij\cdot} - \bar{X}_{\cdot j\cdot})(\bar{X}_{it\cdot} - \bar{X}_{\cdot t\cdot})$$

is useful in finding an unbiased estimator for V_{djt}. The expected value of this statistic is derived below.

$$E\left[\sum_{i=1}^{b} n_{ij}\, n_{it}\, (\bar{X}_{ij\cdot} - \bar{X}_{\cdot j\cdot})(\bar{X}_{it\cdot} - \bar{X}_{\cdot t\cdot})\right]$$

$$= E\left[\sum_{i=1}^{b} n_{ij}\, n_{it}\, (\bar{X}_{ij\cdot}\, \bar{X}_{it\cdot} - \bar{X}_{ij\cdot}\, \bar{X}_{\cdot t\cdot} - \bar{X}_{\cdot j\cdot}\, \bar{X}_{it\cdot} + \bar{X}_{\cdot j\cdot}\, \bar{X}_{\cdot t\cdot})\right]$$

After substituting A.43, A.58, A.59, A.60, A.64, and A.65,

$$E\left[\sum_{i=1}^{b} n_{ij}\, n_{it}\,(\overline{X}_{ij\cdot} - \overline{X}_{\cdot j\cdot})\,(\overline{X}_{it\cdot} - \overline{X}_{\cdot t\cdot})\right]$$

$$= \sum_{i=1}^{b} n_{ij}\, n_{it}\left((\mu + A_j)(\mu + A_t) + \frac{1}{2}(V_{bj} + V_{bt} - V_{djt}) + \frac{V'_{rjt}}{\max(n_{ij}, n_{it})}\right)$$

$$-2 \sum_{i=1}^{b} n_{ij}\, n_{it}\left((\mu + A_j)(\mu + A_t) + \frac{1}{2b_{jt}}(V_{bj} + V_{bt} - V_{djt}) + \frac{V'_{rjt}}{b_{jt}\max(n_{ij}, n_{it})}\right)$$

$$+ \sum_{i=1}^{b} n_{ij}\, n_{it}\Bigg((\mu + A_j)(\mu + A_t) + \frac{1}{2b_{jt}}(V_{bj} + V_{bt} - V_{djt}) + \frac{1}{b_{jt}^2}\sum_{\substack{s,\\ n_{sj}\cdot n_{st}>0}} \frac{V'_{rjt}}{\max(n_{sj}, n_{st})}\Bigg)$$

$$= \left(\frac{b_{jt} - 1}{2b_{jt}}\right)\left(\sum_{i=1}^{b} n_{ij}\, n_{it}\right)(V_{bj} + V_{bt} - V_{djt})$$

$$+ V'_{rjt}\left(\left(\frac{b_{jt}-2}{b_{jt}}\right)\sum_{i=1}^{b} \min(n_{ij}, n_{it}) + \frac{1}{b_{jt}^2}\left(\sum_{i=1}^{b} n_{ij}\cdot n_{it}\right)\left(\sum_{\substack{s,\\ n_{sj}\cdot n_{st}>0}} \frac{1}{\max(n_{sj}, n_{st})}\right)\right)$$

for $b_j = b_t = b_{jt}$

(A.68)

The desired estimator for V_{djt} is found by solving A.68 for V_{djt} and replacing V'_{rjt}, V_{bj} and V_{bt} with their respective estimators. Thus,

$$\hat{V}_{djt} = \hat{V}_{bj} + \hat{V}_{bt} - \frac{2b_{jt}}{(b_{jt}-1)(\sum_{i=1}^{b} n_{ij}n_{it})} \sum_{i=1}^{b} n_{ij}\, n_{it}\, (\bar{X}_{ij\cdot} - \bar{X}_{\cdot j\cdot})(\bar{X}_{it\cdot} - \bar{X}_{\cdot t\cdot})$$

$$+ 2V'_{rjt}\left(\left(\frac{b_{jt}-2}{b_{jt}-1}\right)\left(\sum_{i=1}^{b} \min(n_{ij}, n_{it})\right) \Big/ \left(\sum_{i=1}^{b} n_{ij}, n_{it}\right)\right.$$

$$\left. + \frac{1}{b_{jt}(b_{jt}-1)}\left(\sum_{\substack{s,\\ n_{sj}\cdot n_{st}>0}} \frac{1}{\max(n_{sj}, n_{st})}\right)\right) ,$$

for $b_j = b_t = b_{jt}$. (A.69)

Using the above equation for $\hat{V}_{djt}$ and $\hat{V}'_{rjt}$ given by A.67, estimates of $V(\bar{X}_{\cdot j\cdot}-\bar{X}_{\cdot t\cdot})$ can be obtained by substituting $\hat{V}_{djt}$ and $\hat{V}'_{rjt}$ into A.63. That is

$$\hat{V}(\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdot t\cdot}) = \frac{1}{b_{jt}} \hat{V}_{djt} + \frac{1}{b_{jt}^2}\hat{V}_{rj} \sum_{\substack{i,\\ n_{ij}>0}} \frac{1}{n_{ij}} + \frac{1}{b_{jt}^2}\hat{V}_{rt} \sum_{\substack{i,\\ n_{it}>0}} \frac{1}{n_{it}}$$

$$- \frac{2\hat{V}'_{rjt}}{b_{jt}^2} \sum_{\substack{i,\\ n_{ij}\cdot n_{it}>0}} \frac{1}{\max(n_{ij}, n_{it})} , \text{ for } b_j = b_t = b_{jt}. \qquad \text{(A.70)}$$

APPENDIX B

LEAST-COST EXPERIMENTAL DESIGN ALGORITHM

This appendix presents the algorithm used to solve for the least-cost experimental design to achieve a set of estimated variances less than a specified upper limit, i.e., V_s, for each system mean estimate. The algorithm is provided a set of estimates for the system block variances, V_{bj}, and system replication variances, V_{rj}. In addition, a set of blocks may already have been observed, and, if so, the algorithm must determine the additional experimentation in the form of additional replications for the old blocks and/or new blocks with replications. The algorithm considers the expected cost to generate a block and to perform replications within a block in order to reduce the estimated variances for each system mean estimate blow the limit V_s. The costs provided as input data to the algorithm are:

C_b = cost to generate a new block, and
C_r = cost to perform a single replication for one system, in addition to C_b.

The solution provided by the algorithm is expressed by the following variables:

b_j^{u*} = total blocks to be observed for system j

b^* = maximum number of blocks to be observed over all systems, i.e., $b^* = \max b_j^{u*}$

n_j^{r*} = minimum number of replications in each block

b' = number of blocks already initiated.

That is, the additional replications for system j to be observed in blocks already initiated is

$$n_j^{r*} - n_{ij} \text{ if } n_j^{r*} - n_{ij} > 0, \text{ and}$$

$$0 \text{ if } n_j^{r*} - n_{ij} \leq 0; \text{ where } i = 1,2,\cdots,b', \text{ and}$$

n_{ij} is defined on page 50. In addition, $b_j^{u*} - b'$ new blocks must be initiated for system j with n_j^{r*} replications in each block.

Of course the algorithm can operate in the absence of blocks that have already been initiated. In this case, b' is zero and $n_{ij} = 0$ for all i.

The algorithm finds a least-cost solution iteratively by determining the conditional least-cost solution given b blocks for all possible values of b. For each conditional least-cost solution, the following values are generated:

b_j^u = total blocks to be observed for system j given a maximum of b blocks ($b_j^u \le b$),

n_j^r = minimum number of replications in each block given a maximum of b blocks.

The lowest possible value of b is b' where all additional replications are performed on existing blocks and the largest possible value of b occurs when $n_j^r = 1$ for all j.

The solution for the conditional least-cost test plan given b blocks makes use of the decomposition permitted into individual system test plans when the number of blocks generated is known. That is, the least-cost test plan from a overall viewpoint consists of the least-cost test plan for each system considered individually to achieve a variance of the system mean estimate no greater than V_s. This decomposition is only permitted when the total number of blocks is known so that tradeoffs between the total number of blocks and replications for each system are not required.

The least cost solution for n_j^r is the lowest value of n_j^r satisfying the constraint giving the upper limit on the estimated variance after the experimental plan is executed. That is, application of A.42 specifies the choice of the smallest value of n_j^r satisfying

$$\frac{V_{bj}}{b} + \frac{1}{b^2} \sum_{i=1}^{b'} \frac{V_{rj}}{\max(n_j^r, n_{ij})} + \frac{(b-b')V_{rj}}{b^2 \cdot n_j^r} \le V_s \tag{B.1}$$

A solution to equation B.1 giving a value of n_j^r is complicated by the summation

$$\sum_{i=1}^{b'} \frac{V_{rj}}{\max(n_j^r, n_{ij})},$$

but the solution is simplified by ranking the values of n_{ij}; $i = 1,2,\cdots, b'$; such that $n_{tj}^* \ge n_{t+1,j}^*$; $t = 1,2,\cdots,b'$; where n_{tj}^* is a ranked value of n_{ij}. Substituting these ranked values, B.1 can be rewritten giving

$$\frac{V_{bj}}{b} + \frac{1}{b^2} \sum_{t=1}^{m} \frac{V_{rj}}{n_{tj}^*} + \frac{(b-m)V_{rj}}{b^2 \cdot n_j^r} \le V_s , \tag{B.2}$$

where m satisfies $n_{mj}^* > n_j^r \ge n_{m+1,j}^*$ and n_j^r is the smallest integer satisfying B.2. Assumimg that m is known, then

$$n_j^r = \left[\frac{(b-m)V_{rj}}{b^2V_s - bV_{bj} - \sum_{t=1}^{m} \frac{V_{rj}}{n_{tj}^*}} \right] + 1 \quad , \tag{B.3}$$

if

$$b^2V_s - bV_{bj} - \sum_{t=1}^{m} \frac{V_{rj}}{n_{tj}^*} > 0 \; ,$$

where m satifies $n_{mj}^* > n_j^r \geq n_{m+1,j}^*$, and the function [X] is the greatest integer less than X. In the event

$$b^2V_s - bV_{bj} - \sum_{t=1}^{m} \frac{V_{rj}}{n_{tj}^*} \leq 0,$$

then n_j^r does not exist and more blocks must be generated. The value of n_j^r is determined iteratively using the procedure flowcharted in Figure B.1.

A special case occurs when the solution for n_j^r by B.3 is one. In this case, the least cost solution may not require a replication in each block, i.e., b_j^u may be less than b. Applying equation B.2, the smallest value of b_j^u satisfying

$$\frac{V_{bj}}{b_j^u} + \frac{1}{\left(b_j^u\right)^2} \sum_{i,\, n_{ij}>0} \frac{V_{rj}}{n_{ij}} + \frac{(b_j^u - b_j)V_{rj}}{(b_j^u)^2} \leq V_s \; , \tag{B.4}$$

$b_j^u \leq b'$, and

b_j = number of blocks already generated with $n_{ij} > 0$,

gives the required conditional least-cost solution when n_j^r determined by B.3 is one.

A flowchart of the algorithm for determining the least cost experimental design appears in Figure B.1. The algorithm generates each possible conditional least cost solution for all permissible values of b. The conditional least cost given b total blocks is computed by the following equation

$$C_{tb} = C_b \cdot b + \sum_{j=1}^{a} n_j^r \cdot C_r(b_j^u - b')$$
$$+ \sum_{j=1}^{a} \sum_{i=1}^{b'} U(n_j^r - n_{ij}) \cdot (n_j^r - n_{ij}) \cdot C_r$$

where

C_{tb} = conditional least cost of achieving the variance constraint given b blocks, and

$U_{(x)}$ = the stop function where $U_{(x)} = \begin{cases} 1 \text{ if } X > 0 \\ 0 \text{ if otherwise.} \end{cases}$

The least cost test plan is determined by finding the value of b^* satisfying $C_{tb*} = \underset{b}{\mathrm{Min}} C_{tb}$; and values of b_j^u and n_j^r, $j = 1,2,\cdots,a$; for this test plan are designated as the least-cost values b_j^{u*} and n_j^{r*}. Each possible value of b is calculated in order to determine b^* since C_{tb} is not a unimodal function of b.

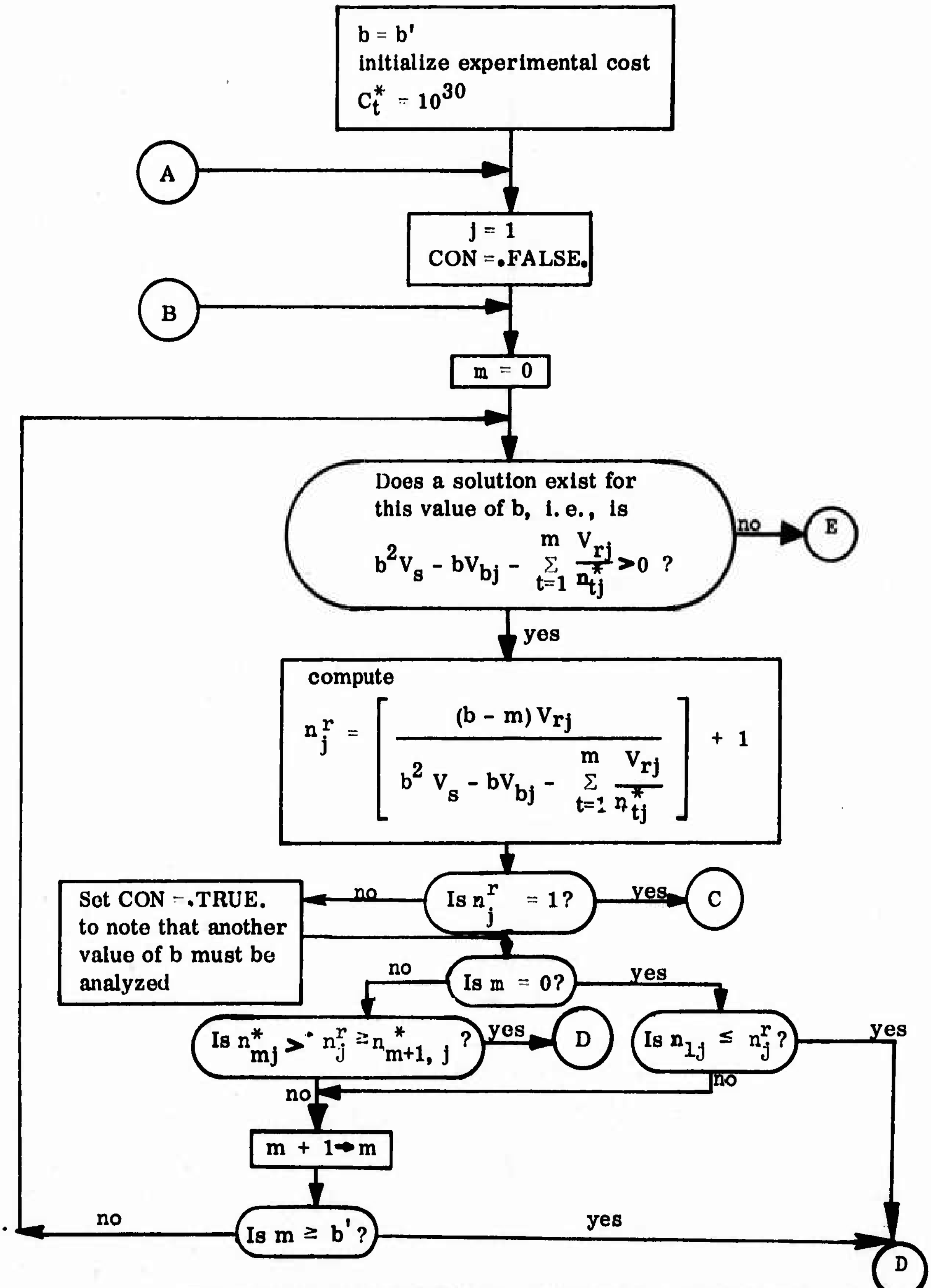

Figure B.1.--Algorithm for Determining Least Cost Solution

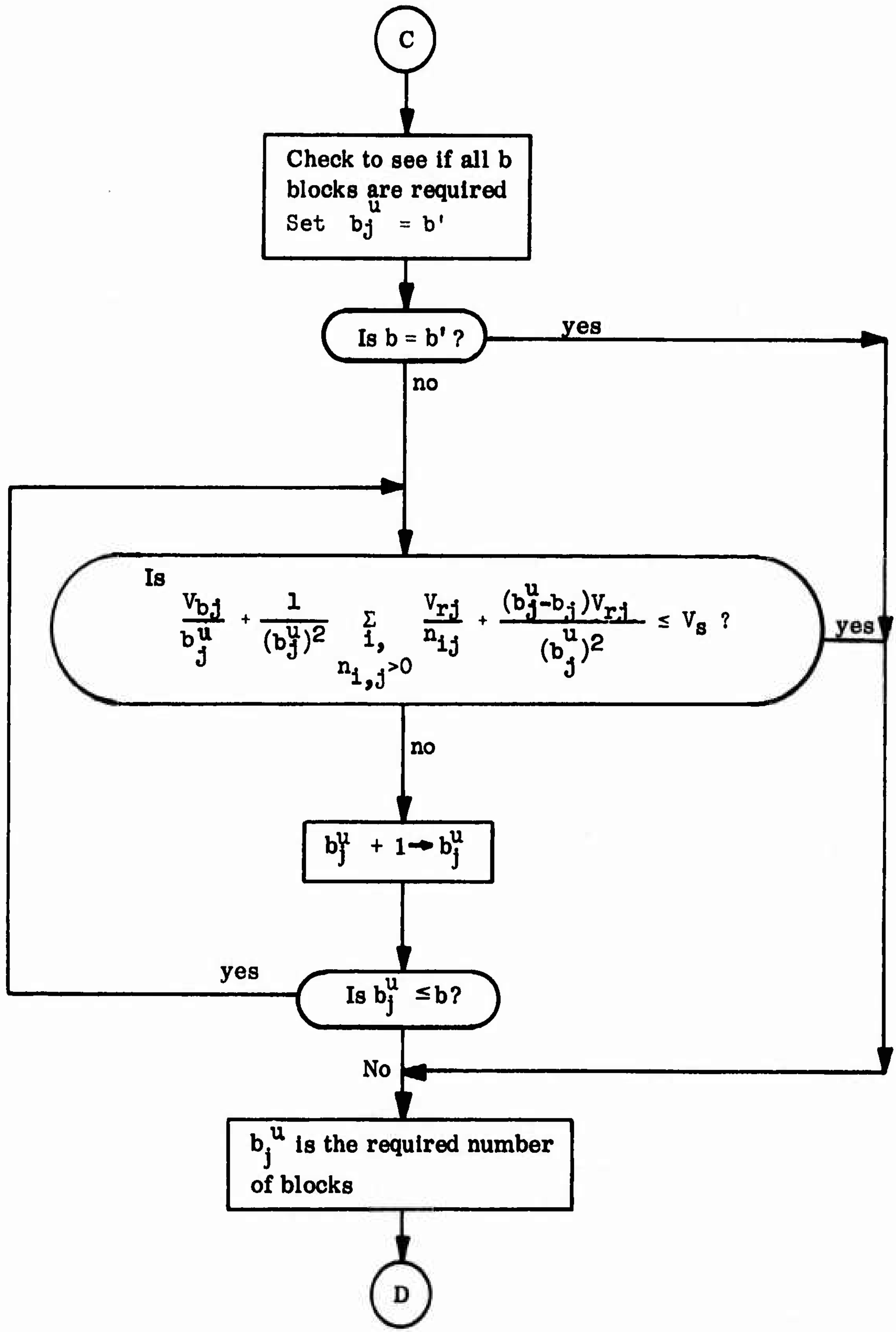

Figure B.1, continued--Algorithm for Determining Least Cost Solution

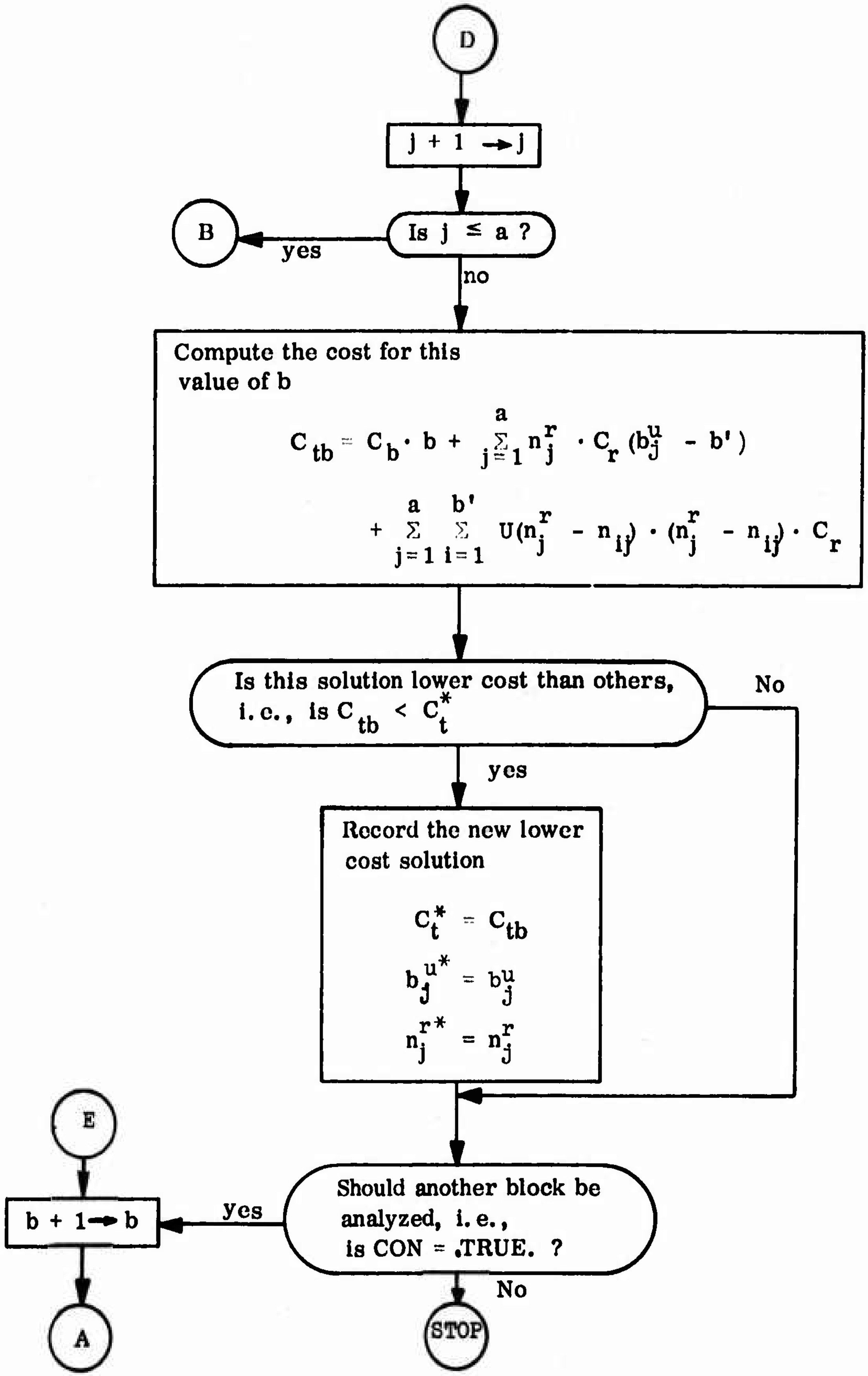

Figure B.1, continued--Algorithm for Determining Least Cost Solution

APPENDIX C

DERIVATION OF ESTIMATORS FOR THE FILTER INDEPENDENT EFFECTS AND HETEROGENEOUS VARIANCES MODEL (HEVIE)

Model Description

A special case in which the component effects of the Filter model derived in Appendix A are independent with heterogeneous variances is presented in this appendix, and estimators for the model's parameters are derived. The model acronym is HEVIE for Heterogeneous Variances and Independent Effects. In this model, the replication effects, ϵ_{ijk}, have unique variances for each system, σ^2_{ej}, as defined in Appendix A, but these replication effects are independent of the block main effects, B_i, and the interaction effects, AB_{ij}. In addition, the block main effects are independent of the interaction effects. Thus, the HEVIE model is a special case of the HEVCE model derived in Appendix A, where

$$\sigma_{BAB} = 0$$

$$\sigma_{ABAB} = 0$$

$$\sigma_{Bej} = 0$$

$$\sigma_{ABej} = 0$$

$$\sigma'_{ABej} = 0$$

$$\sigma'_{ej} = 0$$

$$\sigma'_{ejt} = 0$$

$$\sigma^*_{ejt} = 0$$

for $j = 1,2,\cdots,a$; and $t = 1,2,\cdots,a$. The parameters remaining in the HEVIE model are σ^2_B, σ^2_{AB}, and σ^2_{ej}, $j = 1,2,\cdots,a$.

The basic purpose of simulation experiments is to estimate the mean system performances, $\bar{X}_{\cdot j \cdot}$, the variance of the estimators, $V(\bar{X}_{\cdot j \cdot})$, and the variances of the differences between two system performances, $V(\bar{X}_{\cdot j \cdot}-\bar{X}_{\cdot t \cdot})$. The expression for $V(\bar{X}_{\cdot j \cdot})$ given by equation A.42 of the HEVCE Model becomes*

$$V(\bar{X}_{\cdot j \cdot}) = \frac{1}{b}\left(\sigma^2_B + \sigma^2_{AB}\right) + \frac{\sigma^2_{ej}}{b^2} \sum_{i=1}^{b} \frac{1}{n_{ij}}\Bigg) \tag{C.1}$$

*In the HEVIE model, the assumption is made that $n_{ij} > 0$ for i and j; thus, $b_j = b$, $j = 1,2,\cdots,a$.

after setting all covariances in A.42 to zero. Thus, the system j block variance for the HEVIE model becomes

$$V_{bj} = \sigma_B^2 + \sigma_{AB}^2, \tag{C.2}$$

and the system j replication variance is

$$V_{rj} = \sigma_{ej}^2 \tag{C.3}$$

for $j = 1,2,\cdots,a$. Note that the block variances are identical for each system in this case. The variance of the difference between system average performances, $V(\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot})$ is simply the sum

$$V(\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdot t \cdot}) = V(\bar{X}_{\cdot j \cdot}) + V(\bar{X}_{\cdot t \cdot}) \tag{C.4}$$

since the covariances all vanish. Thus, the HEVIE model can be implemented if estimators of V_{bj} and V_{rj}, $j = 1,2,\cdots,a$, are obtained. Moreover, these estimators can be used in the procedure described in Appendix B to find the least cost test plan.

Estimators

Replication Variances

Equation A.47 can be used to obtain an estimator for V_{rj} which is identical to the estimator employed by the HEVCE model. Thus,

$$\hat{V}_{rj} = \left[1/\left(\sum_{i=1}^{b} n_{ij} - b\right)\right]\left[\sum_{i=1}^{b}\sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij\cdot})^2\right]$$

for $j = 1,2,\cdots,a$. (C.5)

Note that the estimator for V_{rj} is unbiased.

Block Variance

The estimator for V_{bj} takes a different form for the HEVIE model than the HEVCE model primarily because the value of V_{bj} is identical for each system. To estimate $V_{bj} = \sigma_B^2 + \sigma_{AB}^2$, the expectations for two statistics not employed in the HEVCE model are required. These statistics are commonly known as the block sum of squares, i.e.,

$\sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2$, and the interaction sum of squares, i.e.,

$$\sum_{i=1}^{b} \sum_{j=1}^{a} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdots})^2.$$

Block Sum of Squares

The expected value of the block sum of squares can be expressed as

$$E\left[\sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdots})^2\right] = E\left[\sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot})^2\right] - b\ E\left[(\bar{X}_{\cdots})^2\right]$$

$$= \sum_{i=1}^{b} E\left[(X_{i\cdot\cdot})^2\right] - b\ E\left[(X_{\cdots})^2\right] \qquad (C.6)$$

The expectation of $\bar{X}_{i\cdot\cdot}{}^2$ is evaluated below.

$$E\left[(\bar{X}_{i\cdot\cdot})^2\right] = E\left[\left[\frac{1}{a}\sum_{j=1}^{a}\frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}}(\mu + A_j + B_i + AB_{ij} + \epsilon_{ijk})\right]^2\right]$$

$$= \frac{1}{a^2}\ E\left[\left(a\,\mu + a\,B_i + \sum_{j=1}^{a}\frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}}\epsilon_{ijk}\right)^2\right]$$

since $\sum_{i=1}^{a} A_j = 0$ and $\sum_{j=1}^{a} AB_{ij} = 0$ as expressed by equations A.2 and A.7.

Continuing,

$$E\left[(\bar{X}_{i\cdot\cdot})^2\right] = \frac{1}{a^2}\ E\left[a^2\mu^2 + a^2 B_i{}^2 + \left(\sum_{j=1}^{a}\frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}}\epsilon_{ijk}\right)^2\right]$$

after deleting all products of a constant and a zero mean random variable, and products of two zero mean independent random variables. Thus,

$$E\left[(\bar{X}_{i\cdot\cdot})^2\right] = \mu^2 + \sigma_B^2 + \frac{1}{a^2}\sum_{j=1}^{a}\frac{\sigma_{ej}^2}{n_{ij}}, \qquad (C.7)$$

since the values of ϵ_{ijk} are mutually independent.

Using the above result for $E[(\bar{X}_{i\cdot\cdot})^2]$, the expectation of $\bar{X}_{\cdot\cdot\cdot}^2$ can be determined.

$$E[\bar{X}_{\cdot\cdot\cdot}^2] = E\left[\left(\frac{1}{b}\sum_{i=1}^{b}\bar{X}_{i\cdot\cdot}\right)^2\right]$$

$$= \frac{1}{b^2}E\left[\sum_{i=1}^{b}\sum_{s=1}^{b}\bar{X}_{i\cdot\cdot}\bar{X}_{s\cdot\cdot}\right] = \frac{1}{b^2}E\left[\sum_{i=1}^{b}\bar{X}_{i\cdot\cdot}^2 + 2\sum_{i=1}^{b}\sum_{s=i+1}^{b}\bar{X}_{i\cdot\cdot}\bar{X}_{s\cdot\cdot}\right]$$

$$= \frac{1}{b}\left(\mu^2 + \sigma_B^2 + \frac{1}{ba^2}\sum_{j=1}^{a}\sum_{i=1}^{b}\frac{\sigma_{ej}^2}{n_{ij}}\right) + \frac{(b-1)}{b}\mu^2, \tag{C.8}$$

where the relationships

$$E[\bar{X}_{i\cdot\cdot}\bar{X}_{s\cdot\cdot}] = \mu^2, \text{ and } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

were used in evaluating the last term in C.8.

Substituting C.7 and C.8 into C.6, the expected value of the block sum of squares becomes

$$E\left[\sum_{i=1}^{b}(\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2\right] = b\left(\mu^2 + \sigma_B^2\right) + \frac{1}{a^2}\sum_{j=1}^{a}\sum_{i=1}^{b}\frac{\sigma_{ej}^2}{n_{ij}}$$

$$- \left(\mu^2 + \sigma_B^2 + \frac{1}{a^2 b}\sum_{j=1}^{a}\sum_{i=1}^{b}\frac{\sigma_{ej}^2}{n_{ij}}\right) - (b-1)\mu^2$$

$$= (b-1)\left(\sigma_B^2 + \frac{1}{a^2 b}\sum_{j=1}^{a}\sum_{i=1}^{b}\frac{\sigma_{ej}^2}{n_{ij}}\right) \tag{C.9}$$

Interaction Sum of Squares

The expected value of the interaction sum of squares is derived below.

$$E\left[\sum_{i=1}^{b}\sum_{j=1}^{a}(\bar{X}_{ij\cdot}-\bar{X}_{i\cdot\cdot}-\bar{X}_{\cdot j\cdot}+\bar{X}_{\cdots})^2\right]$$

$$= E\left[\sum_{i=1}^{b}\sum_{j=1}^{a}(\bar{X}_{ij\cdot}^2+\bar{X}_{i\cdot\cdot}^2+\bar{X}_{\cdot j\cdot}^2+\bar{X}_{\cdots}^2-2\bar{X}_{ij\cdot}\,\bar{X}_{i\cdot\cdot}-2\bar{X}_{ij\cdot}\,\bar{X}_{\cdot j\cdot}\right.$$

$$\left.+2\bar{X}_{ij\cdot}\,\bar{X}_{\cdots}+2\bar{X}_{i\cdot\cdot}\,\bar{X}_{\cdot j\cdot}-2\bar{X}_{i\cdot\cdot}\,\bar{X}_{\cdots}-2\bar{X}_{\cdot j\cdot}\,\bar{X}_{\cdots})\right]$$

after expanding the expression squared,

$$= E\left[\sum_{i=1}^{b}\sum_{j=1}^{a}\bar{X}_{ij\cdot}^2\right]+E\left[\sum_{i=1}^{b}a\,\bar{X}_{i\cdot\cdot}^2\right]+E\left[\sum_{j=1}^{a}b\,\bar{X}_{\cdot j\cdot}^2\right]+ab\,E\left[\bar{X}_{\cdots}^2\right]$$

$$+E\left[\sum_{i=1}^{b}(-2a\,\bar{X}_{i\cdot\cdot}^2+2a\,\bar{X}_{i\cdot\cdot}\,\bar{X}_{\cdots}+2a\,\bar{X}_{i\cdot\cdot}\,\bar{X}_{\cdots}-2a\,\bar{X}_{\cdots}^2)\right]$$

$$+E\left[\sum_{j=1}^{a}(-2b\,\bar{X}_{\cdot j\cdot}^2-2b\,\bar{X}_{\cdots}^2)\right]$$

$$= E\left[\sum_{i=1}^{b}\sum_{j=1}^{a}\bar{X}_{ij\cdot}^2\right]-a\,E\left[\sum_{i=1}^{b}\bar{X}_{i\cdot\cdot}^2\right]-b\,E\left[\sum_{j=1}^{a}\bar{X}_{\cdot j\cdot}^2\right]+ab\,E\left[\bar{X}_{\cdots}^2\right] \tag{C.10}$$

Using equation A.41 and noting that

$$\sigma_{BAB}=\sigma_{Bej}=\sigma_{ABej}=\sigma'_{ej}=0$$

for the HEVIE model, then

$$E\,[\bar{X}_{ij\cdot}^2\,]=(\mu+A_j)^2+\sigma_B^2+\sigma_{AB}^2+\frac{\sigma_{ej}^2}{n_{ij}}\,. \tag{C.11}$$

Similarly from equation A.50, we obtain

$$E\,[\,\bar{X}_{\cdot j\cdot}^2\,]=(\mu+A_j)^2+\frac{1}{b}(\sigma_B^2+\sigma_{AB}^2)+\frac{1}{b^2}\sigma_{ej}^2\sum_{i=1}^{b}\frac{1}{n_{ij}} \tag{C.12}$$

In using the above equations to evaluate

$$E\left[\sum_{j=1}^{a} \bar{X}_{ij\cdot}^{2}\right] \text{ and } E\left[\sum_{j=1}^{a} \bar{X}_{\cdot j\cdot}^{2}\right], \text{ the constraint } \sum_{j=1}^{a} AB_{ij} = 0 \text{ should}$$

be recognized.

In evaluating $E\left[\sum_{j=1}^{a} AB_{ij}^{2}\right]$ a degree of freedom is lost to the constraint so that

$$E\left[\sum_{j=1}^{a} AB_{ij}^{2}\right] = (a-1)\,\sigma_{AB}^{2}\,.$$

Substituting these relations into C.10, we have

$$E\left[\sum_{i=1}^{b}\sum_{j=1}^{a} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^{2}\right]$$

$$= \sum_{j=1}^{a} b\,(\mu + A_j)^2 + ab\sigma_B^2 + (a-1)\,b\,\sigma_{AB}^2 + \sum_{j=1}^{a} \sigma_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}}$$

$$- a\,b\,(\mu^2 + \sigma_B^2) - \frac{1}{a}\sum_{j=1}^{a} \sigma_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}} - \sum_{j=1}^{a} b\,(\mu + A_j)^2$$

$$- a\,\sigma_B^2 - (a-1)\,\sigma_{AB}^2 - \frac{1}{b}\sum_{j=1}^{a} \sigma_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}}$$

$$+ a\,(\mu^2 + \sigma_B^2) + \frac{1}{ab}\sum_{j=1}^{a} \sigma_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}} + a(b-1)\,\mu^2,$$

$$= (a-1)\,(b-1)\,\sigma_{AB}^2 + \frac{(a-1)(b-1)}{ab} \sum_{j=1}^{a} \sigma_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}} \tag{C.13}$$

<u>Block Variance Estimator</u>

The expected value expressions for the block and interaction sum of squares derived above provide the basis for an unbiased estimator of the block variance $V_{bj} = \sigma_B^2 + \sigma_{AB}^2$. Using C.13 and C.9, we can solve for the expected value of V_{bj}. After replacing parameter values with unbiased estimators and removing expectation symbols, then

$$\hat{V}_{bj} = \hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2 = \sum_{i=1}^{b} (\bar{X}_{i..} - \bar{X}_{...})^2 / (b - 1)$$

$$+ \sum_{i=1}^{b} \sum_{j=1}^{a} (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...})^2/[(a-1)(b-1)]$$

$$- \frac{(a-1)}{a^2 b} \sum_{j=1}^{a} \hat{\sigma}_{ej}^2 \sum_{i=1}^{b} \frac{1}{n_{ij}} \qquad (C.14)$$

Note that C.14 is an unbiased estimator for V_{bj}; however, negative estimates may occur in practice for $\hat{V}_{bj}$ when the estimates for V_{rj} are sufficiently large and the interaction and block sum of squares are sufficiently small. Of course, a negative value of $\hat{V}_{bj}$ has no practical meaning, so a procedure similar to that employed by the HEVCE model is used. Thus, $\hat{V}_{bj}$ will be used from C.14 if it is positive; otherwise, $\hat{V}_{bj}$ is set to zero. The assumption is made that the true value of V_{bj} is likely to be negligible when negative values from C.14 occur. Although this procedure will yield better estimates, the estimator for $\hat{V}_{bj}$ that deletes negative values is biased.

APPENDIX D

DERIVATION OF ESTIMATORS FOR THE FILTER INDEPENDENT EFFECTS AND HOMOGENEOUS VARIANCES MODEL (HOVIE)

Model Description

This appendix presents the Filter Independent Effects and Homogeneous Variances Model (HOVIE), and estimators for model parameters are derived. The HOVIE model amounts to a further simplification of the HEVIE model presented in Appendix C, and the HOVIE model is equivalent to mixed two-factor model commonly found in experimental design literature (Hicks, 1964; Graybill, 1961; and Winer, 1971). The HOVIE model is obtained from the HEVIE model by setting

$$\sigma^2_{ej} = \sigma^2_e \text{ for all } j. \tag{D.1}$$

Thus, the HOVIE model specifies that all systems have the same variance, and it follows that they would have the same test plan.

The variance of the estimator for system j mean performance is

$$V(\bar{X}_{\cdot j \cdot}) = \frac{1}{b}(\sigma^2_B + \sigma^2_{AB}) + \frac{\sigma^2_e}{b^2} \sum_{i=1}^{b} \frac{1}{n_{ij}} \tag{D.2}$$

which is obtained directly from C.1. As specified for the HEVIE model by C.2, the system j block variance for the HOVIE model is

$$V_{bj} = \sigma^2_B + \sigma^2_{AB} \tag{D.3}$$

In addition, the replication variance is

$$V_{rj} = \sigma^2_e, \text{ for } j = 1,2,\cdots,a. \tag{D.4}$$

As done by the other models, the least-cost experimental design can be prepared given estimates of V_{bj} and V_{rj}, and these estimates can be used in the procedure described in Appendix B to prepare the experimental plan.

Parameter Estimators

Estimators for V_{rj} and V_{bj} are specified in this section using results developed for the HEVIE and HEVCE models.

Both the HEVIE and HEVCE models use the result specified by equation A.46 in deriving an unbiased estimator for V_{rj}. That is,

$$E\left[\sum_{\substack{i,\\ n_{ij}>0}} \sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij\cdot})^2\right] = V_{rj}\left[\sum_{\substack{i,\\ n_{ij}>0}} n_{ij} - b_j\right]$$

Since the replication variances for each system are identical for this model, then the statistic cited above can be summed over each system to provide an estimator for $V_{rj} = \sigma_e^2$; $j = 1,2,\cdots,a$. The expectation of this statistic is*

$$E\left[\sum_{i=1}^{b} \sum_{j=1}^{a} \sum_{k=1}^{n_{ij}} (X_{ijk} - X_{ij\cdot})^2\right] = \left[\sum_{i=1}^{b} \sum_{j=1}^{a} n_{ij} - ab\right] V_{rj} \tag{D.5}$$

Thus, an unbiased estimator for V_{rj} is

$$\hat{V}_{rj} = \sum_{i=1}^{b} \sum_{j=1}^{a} \sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij\cdot})^2 \Big/ \left[\sum_{i=1}^{b} \sum_{k=1}^{a} n_{ij} - ab\right] = \hat{\sigma}_e^2 . \tag{D.6}$$

The same estimator as employed by the HEVIE model can be used for $\hat{V}_{bj}$ after accounting for the fact that $\hat{V}_{rj}$ is a constant function of j. Thus,

$$\hat{V}_{bj} = \hat{\sigma}_B^2 + \hat{\sigma}_{AB}^2 = \sum_{i=1}^{b} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2/(b-1)$$

$$+ \sum_{i=1}^{b} \sum_{j=1}^{a} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^2/[(a-1)(b-1)]$$

$$- \frac{(a+1)}{a^2 b} \hat{\sigma}_e^2 \sum_{j=1}^{a} \sum_{i=1}^{b} \frac{1}{n_{ij}} \tag{D.7}$$

*All values of n_{ij} are assumed to be greater than zero in the HOVIE model.

D.7 is an unbiased estimator for V_{bj}, but again, adjustments may have to be made in the event $\hat{V}_{bj}$ is less than zero. As specified for the HEVIE and HEVCE models, negative values of $\hat{V}_{bj}$ are set to zero giving more realistic estimates but removing the unbiased property for $\hat{V}_{bj}$.

APPENDIX E

FILTER PROGRAM

Description

Purpose

The Filter program accepts data generated by blocked simulation replications and estimates Filter Model parameters. These parameters are used to estimate the variance of average system performance for each system alternative and the variance of the difference in average performance between each possible pair of system alternatives. Finally, a set of least-cost test plans are determined so that the variance of each system's average performance is less than a specified upper limit V_s. A sequence of values for V_s are considered, an a test plan is determined for value of V_s.

Input Data Format

Card Number	Variable	Columns	Format
1	Title	1 to 60	Literal characters
2	Title, continued	1 to 60	Literal characters
3	NS	1 to 5	integer
3	BP	6 to 10	integer
3	NVARS	11 to 15	integer
3	VARI	16 to 25	real
3	DVARS	26 to 35	real
3	CR	36 to 45	real
3	CB	46 to 55	real
4	N(1,1)	1 to 5	integer
4	N(1,2)	6 to 10	integer
.	.	.	.
.	.	.	.
.	.	.	.
4	N(1,NS)	4NS + 1 to 5NS	integer
4	N(2,1)	5NS + 1 to 5 (NS+1)	integer
.	.	.	.
.	.	.	.
.	.	.	.
3 + NN	N(BP,NS)	.	integer

(Continue entering the N array by rows starting with the first row, then the second row. Each value is entered in a field of length five columns, and NN cards are used with eighty columns available on each card.)

Card Number	Variable	Columns	Format
4+NN	X(1,1,1)	1 to 10	real
4+NN	X(1,1,2)	11 to 20	real
4+NN	X(1,1,3)	21 to 30	real
.	.	.	.
.	.	.	.
.	.	.	.
4+NN	X(1,1,N(1,1))	10 N(1,1)-9 to 10 N(1,1)	real
5+NN	X(1,2,1)	1 to 10	real
5+NN	X(1,2,2)	11 to 20	real

(Continue entering observation values for the first block and second system. Up to eight entries can be made on a single card. If more than eight observations have been recorded for a system, use more than one card. Always start a system on a new card. After recording the first block, enter the observations for the second block using the same format.)

Definition of Variables

BJ(J)	=	Number of blocks having replications of system J
BP	=	Number of blocks already run
BU(J)	=	Total blocks to be run for system J
CB	=	Cost to run a block
CR	=	Cost to perform one replication of one system
DVARS	=	Decrease in value of V_s for each case considered
N(I,J)	=	Number of replications already run for system J in block I
NR(J)	=	Number of replications for system J to be run in each block
NS	=	Number of systems
NVARS	=	Number of different values of V_s considered (number of different test plans computed)
VARI	=	Initial value of V_s considered (each successive value is decreased by DVARS)
VARS	=	Upper limit on variance of system
VBE(J)	=	System J block variance ($\hat{V}_{bj}$)
VDBE(J,K)	=	Block effect variance for estimated difference between performance of systems J an K ($\hat{V}_{djt}$)
VDPS(J,K)	=	Replication effect variance for the difference between J and K ($\hat{V}'_{rjk}$)
VR(J)	=	System J replication effect variance
VXBAR(J)	=	Estimated variance of the mean performance of system J ($\hat{V}(\overline{X}_{\cdot j \cdot})$)
VXBDIF	=	Variance of the estimated difference between the performance of J and K($\hat{V}(\overline{X}_{\cdot j \cdot}-\overline{X}_{\cdot k \cdot})$)
XBJJ	=	Estimated difference between the mean performance of system J and JJ

Method

1. Read NS, BP, NVARS, VARI, DVARS, CR, CB
   N(I, J) for J = 1, NS and I = 1, BP
   X(I, J, K) for k = 1, N(I, J), J = 1, NS, and I = 1, BP

2. Compute $XJ(J) = \bar{X}_{\cdot j}$. for J = 1, NS

3. Compute

   VR(J) by equation A.47
   VBE(J) by equation A.52
   VXBAR(J) by equation A.53

   for J = 1, NS

4. Compute

   VDPS(J, JJ) by equation A.67
   VDBE(J, JJ) by equation A.69
   VXBDIF by equation A.70

   for J = 1, NS,
   JJ = 1, NS, and
   $J \neq JJ$

5. Set VARS = VARI
   IV = 1

6. Determine the least cost test plan to reduce the estimated variances for each system performance measure below VARS. Use the procedure specified in Appendix B.

7. If IV = NVARS, go to step 9

8. IV = IV + 1
   VARS = VARS - DVARS
   Go to step 6

9. Stop.

References

Booz-Allen Applied Research, A Summary of the Operational Characteristics of FILTER, Combined Arms Research Office, Ft. Leavenworth, Kansas, unpublished report, 1967.

Emshoff and Sisson, Design and Use of Computer Simulation Models, New York: The Macmillan Co., 1970, p. 197.

Fishman, George S., Concepts and Methods in Discrete Event Simulation, New York: John Wiley & Sons, 1973, p. 326.

Graybill, Franklin A., An Introduction to Linear Statistical Models, New York: McGraw-Hill Book Company, Inc., 1961.

Hicks, Charles R., Fundamental Concepts in the Design of Experiments, New York: Holt, Rinehart, and Winston, Inc., 1964.

Hocking, R. R., "A Discussion of the Two-Way Mixed Model," The American Statistician, October 1973, Vol 27, No. 4, pp 148-152.

Scheffe, Henry, The Analysis of Variance, New York: John Wiley & Sons, Inc., 1959.

USACDC, "The IUA Simulation Model," TATAWS III project, unpublished working paper, Fort Knox, Kentucky, 1968.

Winer, B. J., Statistical Principles in Experimental Design, Second Edition, New York: McGraw-Hill Book Company, 1971.